

The University of Chicago  
FOUNDED BY JOHN D. ROCKEFELLER

Concerning a Certain Type of Continued Fractions  
Depending on a Variable Parameter

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE  
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

THOMAS E. McKINNEY

The Lord Baltimore Press  
BALTIMORE, MD., U. S. A.

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# Concerning a Certain Type of Continued Fractions Depending on a Variable Parameter.

BY THOMAS E. MCKINNEY.

## PART I.

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### *Introduction.*

Every system of continued fractions is based on a defining equation related to a partitioned field with each part of which and as its representative is associated a particular number.

The field may be any provided it be precisely defined, *i. e.* it shall be possible to determine by means of a finite number of operations if any given number belongs to the field. The manner of partitioning is subject to the single limitation that every number of the field shall appear in one and only one part. The representative number associated with any part may be any number of the field; the relation, however, of representative number and corresponding part is unique. The defining equation relates any number of the field and its representative number uniquely. Otherwise expressed, the given number and its representative in conjunction with the defining equation serves to define without ambiguity a third number of the field.



In the process of developing a number into a continued fraction it is conceivable that the parts and their corresponding representatives and the defining equations, any or all, may change. It is sufficient if at every stage of this process these elements are determined without ambiguity and in accordance with the restrictions imposed in the preceding paragraph.

Every system of continued fractions thus defined assigns to every number of the field a definite, determinate continued fraction. The effectiveness of a system so defined depends when the field is given largely on the particular choice made of the parts, representative numbers and defining equations with respect to the ease with which the operations involved may be carried on and with respect to the simplicity of form in which the results may be stated.

A satisfactory and complete characterization of the defining equation, the field, the parts, and the representative numbers is the first fundamental problem in every system of continued fractions. A second fundamental problem arises in the attempt to define the numbers of the field in terms of the representative numbers, *i. e.* to assign to every number of the field a definite continued fraction. It is, perhaps, loosely analogous to a fundamental problem in function theory, *viz.*, to define the numbers of a system in terms of numbers of an assigned character.

An important problem in the theory of quadratic forms is related to the continued fractions arising when in a continued fraction in a given system the partial denominators, barring the first, are taken in reverse order.

More fundamental is the grouping of the numbers of the field with respect to the properties of their developments; as, with respect to ultimate identity as in equivalence, with respect to periodicity, or, more generally, with respect to the properties which serve to characterize the numbers of the field as algebraic or transcendental.

The system of continued fractions studied in this paper refers to the field of real numbers. This field is partitioned by means of a parameter  $\lambda$ , the parts being termed intervals. The role played by this parameter suggests the name  $\lambda$ -developments for the continued fractions of this system. This parameter not only affords the means of giving a wide scope to the investigation, but also throws light on certain fundamental phenomena which appear only in a subdued form in the systems hitherto studied so that their real significance has not been generally remarked.

The element of symmetry has been introduced by a suitable disposition of



the intervals in the positive and negative portions of the field, — so that given the  $\lambda$ -development ( $\lambda = \lambda'$ ) of a positive number the like development of the corresponding negative number may be immediately derived. The representative numbers are the integers lying in the corresponding intervals. This holds without exception, though an apparent exception arises when in the process of developing a number into a continued fraction it becomes necessary to subdivide the interval. In this case the representative number of the partial or special interval is the integer lying in the corresponding normal interval, whether that integer be within or without the special interval.

With respect to genetic relationship this investigation has its origin in a paper by Hurwitz.\* The system of  $\lambda$ -developments here considered is a direct generalization of Hurwitz "Die Kettenbruch-Entwicklung zweiter Art," hereafter referred to as  $H_{II}$ . Save the lack of symmetry in the interval system "Die Kettenbruch-Entwicklung erster Art," or  $H_I$ , is to be classed with  $H_{II}$ . The plan followed is substantially that of Hurwitz. His methods have been employed and his results incorporated whenever they served the purposes of this paper. Acknowledgment is made in foot notes. But the nature of the indebtedness is such that a precise formulation and acknowledgment in all cases has not seemed practicable.

For this system the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $\lambda'$  is fundamental. It is determined directly by means of the interval system and the defining equation. On this  $\lambda$ -development taken as known is based the characterization of the  $\lambda$ -development ( $\lambda = \lambda'$ ) of every other real number. A study of the  $\lambda$ -developments ( $\lambda = \lambda'$ ) of certain simple functions of  $\lambda'$  is made in §2 preliminary to the treatment of the general case of any number of the field.

In §3 a finite  $\lambda$ -development ( $\lambda = \lambda'$ ) is assigned to every real number and a criterion is set up for deciding whether a given finite continued fraction equal numerically to a given number is the  $\lambda$ -development ( $\lambda = \lambda'$ ) of that number.

In the characterization of infinite  $\lambda$ -developments ( $\lambda = \lambda'$ ) it has been necessary (§4) to study the question of the convergence of these developments. It is shown that every infinite  $\lambda$ -development ( $\lambda = \lambda'$ ) is convergent and that with exceptions completely specified every infinite  $\lambda$ -sequence ( $\lambda = \lambda'$ ) defines a number of which it is the  $\lambda$ -development ( $\lambda = \lambda'$ ).

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\* Hurwitz: Ueber eine Besondere Art der Kettenbruch-Entwicklung Reeller Grössen, *Acta Mathematica*, Vol. 12.

Out of the study of the denominators of successive convergents of  $H_I$  emerged  $H_{II}$  as the reverse development to  $H_I$ . With an exception specified by Hurwitz the relation is reciprocal. In the general case of reverse  $\lambda$ -developments ( $\lambda = \lambda'$ ) aside from the instances remarked by Hurwitz there is but one other in the continuous infinity of systems here studied and this with an exception—for which reversion in all its generality holds true.

A comparison is made in §6 of several different systems dependent on a parameter  $\lambda$ , so that the results of one system may be transferred in certain cases to another system without the labor of an investigation *ab initio*.

In §7 the relation of  $\lambda$ -developments ( $\lambda = \lambda'$ ) of numbers properly equivalent is investigated. The result is to divide equivalent numbers into two classes: 1.° Equivalent numbers whose  $\lambda$ -developments ( $\lambda = \lambda'$ ) are ultimately alike; 2.° Equivalent numbers whose  $\lambda$ -developments ( $\lambda = \lambda'$ ) are not ultimately alike, but are so related that the  $\lambda$ -developments ( $\lambda = \lambda'$ ) of the one from elements of a certain rank on may be transformed into a corresponding position of the  $\lambda$ -development ( $\lambda = \lambda'$ ) of the other by a very simple rule. The obvious extension of these results to improper equivalence closes the section.

### §1. *Notation and Defining Equations.*

To explain the notation\* of this paper let  $f$  and  $g$  be any real numbers  $f < g$ . Then the expression

$$x = f \dots g \tag{1}$$

is equivalent to

$$f \leq x \leq g.$$

When, however, the greater number comes first, as

$$x = g \dots f, \tag{2}$$

the interpretation is

$$x > g, \text{ or } x < f.$$

We say that  $f$  and  $g$  determine an interval and, indeed, are its limits. To indicate that the relation of equality does not exist between  $x$  and a limit number  $f$ , or  $g$ , *i. e.* to exclude this number from the interval it is inclosed in

---

\* Hurwitz, *loc. cit.*



parentheses. Thus, the expression, or as we may sometimes say, the interval equations

$$\begin{aligned} x &= (f) \dots g, \\ x &= f \dots (g), \quad f < g, \end{aligned}$$

are equivalent, respectively, to

$$\begin{aligned} f &< x \leq g, \\ f &\leq x < g. \end{aligned}$$

From the definition of the symbolic equation (1) it follows that when  $k$  is any real number

$$x + k = f + k \dots g + k \quad (3)$$

$$kx = kf \dots kg, \quad k > 0, \quad (4)$$

$$kx = kg \dots kf, \quad k < 0, \quad (5)$$

$$\frac{k}{x} = \frac{k}{g} \dots \frac{k}{f}, \quad k > 0, \quad (6)$$

$$\frac{k}{x} = \frac{k}{f} \dots \frac{k}{g}, \quad k < 0. \quad (7)$$

From equations (4), (5), (6), (7), are derived the following

$$x = f \dots g, \quad (8)$$

$$-x = -g \dots -f, \quad (9)$$

$$\frac{1}{x} = \frac{1}{g} \dots \frac{1}{f}, \quad (10)$$

$$-\frac{1}{x} = -\frac{1}{f} \dots -\frac{1}{g}. \quad (11)$$

An analogous system of equations follows from the definition of equation (2). When

$$x = f \dots g \quad (12)$$

an interval not containing zero, and

$$y = h \dots l, \quad (13)$$

an interval not containing infinity then for

$$\begin{aligned} |f| + |g| &> |h| + |l| \\ x + y &= f + h \dots g + l. \end{aligned} \quad (15)$$

2.° Adopting the usual representation of real numbers by points on a right line, in fact, a closed line passing through infinity so that  $+\infty$  and  $-\infty$  are represented by the same point, we may interpret equations (1) and (2)

geometrically. The positive direction on the line is that of increasing numbers, in accordance with the usual convention, from left to right. Equation (1), or (2) now defines that interval which lies on the positive side of the first limit number, *i. e.* for equation (1) to the positive side of  $f$  and for equation (2) to the positive side of  $g$ .

In the immediate sequel brevity of statement will require the Kronecker symbol  $\text{sgn. } x$ . This symbol is defined by the equation

$$\text{sgn. } x = \frac{x}{|x|} = \pm 1.$$

where  $x$  is real and not zero. When occasion requires a value will be assigned arbitrarily to  $\text{sgn. } x$ ,  $x = 0$ .

3.° It will facilitate the treatment of the general case to have a notation which changes readily the order of the limit numbers. To do this we inclose the interval limits in parentheses with the proper sign prefixed. Thus:

$$\text{sgn. } x \{a \dots b\} = a \dots b$$

when  $\text{sgn. } x = +1$ , and

$$\text{sgn. } x \{a \dots b\} = -b \dots -a$$

when  $\text{sgn. } x = -1$ .

We shall have occasion to refer to the interval limits  $a$  and  $b$ .  $a$  will be called the left interval limit and  $b$  the right interval limit.

The notation of this paragraph will be employed chiefly in section 3.

4.° Now divide the series of real numbers into the intervals

$$\begin{aligned} \dots; (-2 - \lambda') \dots -1 - \lambda'; (-1 - \lambda') \dots -\lambda'; (-\lambda') \\ \dots (+\lambda'); \lambda' \dots (1 + \lambda'); 1 + \lambda' \dots (2 + \lambda'); \dots, \end{aligned} \quad (15)$$

where  $\lambda'$  is defined by the interval equation

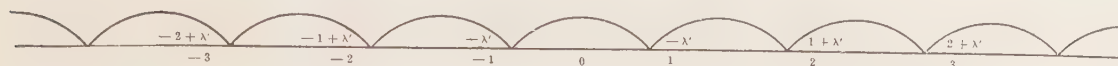
$$\lambda' = (0) \dots 1. \quad (16)$$

Every real number lies in one and only one of these intervals and every interval contains one and only one integer which is called the representative number or integer of the interval. When  $x$  lies in the interval of which  $a$  is the representative number we say that  $a$  represents  $x$ , or that  $x$  corresponds to  $a$  and write

$$x \sim a.$$



The system of intervals (15) may be exhibited graphically on a right line, thus:



The interval limits are written in the intervals to which they belong.

5.° Let  $x_0$  be any real non-zero number and on the basis of the interval system (15) form successively the equations

$$x_0 = a_0 - \frac{1}{x_1}, \quad x_1 = a_1 - \frac{1}{x_2}, \quad x_2 = a_2 - \frac{1}{x_3}, \quad \dots, \quad x_{n-1} = a_{n-1} - \frac{1}{x_n}, \quad (17)$$

where

$$x_i \sim a_i, \quad i = 0, 1, 2, \dots, n-1. \quad (18)$$

Eliminating  $x_1, x_2, \dots, x_{n-1}$  from these equations the results may be expressed as a continued fraction

$$x_0 = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \dots - \frac{1}{a_{n-1} - \frac{1}{x_n}}}}$$

or, according to the notation we shall adopt

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n). \quad (19)$$

Continued fractions of the type (19) whose interval system (15) depends on a parameter  $\lambda$  we call for brevity  $\lambda$ -developments, and when we wish to specify the particular value of the parameter  $\lambda$  on which the interval system depends we shall write  $\lambda$ -developments ( $\lambda = \lambda'$ ). In §6 a fuller notation will be used but there is no occasion to explain it at this point.

When it is desired to assert merely numerical equality without affirming that the second member of the equation is the  $\lambda$ -development ( $\lambda = \lambda'$ ) of the first we shall write

$$x_0 = (a_0, a_1, \dots, a_{n-1}, x_n). \quad (20)$$

When the distinction between partial and complete quotients is not important  $a_0, a_1, \dots, a_{n-1}, x_n$  will be referred to as elements\* of the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $x_0$ .

For a given  $x_0$  and a given  $\lambda$ , say  $\lambda'$ , the elements  $a_0, a_1, \dots, a_{n-1}, x_n$  are uniquely determined so that the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $x_0$  is unique.

\* Dirichlet, Zahlentheorie, §§ 23, 80.

§2.  $\lambda$ -Developments ( $\lambda = \lambda'$ ,  $0 < \lambda' \leq 1$ ) of Certain Functions of  $\lambda'$ .

1.° In this section as a basis\* for the treatment of the more general case of any real number the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $\lambda'$  and of certain simple functions of  $\lambda'$  are derived. The  $\lambda$ -development ( $\lambda = \lambda'$ ) of every real number is limited in a very real, though untechnical, sense by the  $\lambda$ -developments ( $\lambda = \lambda'$ ) of two of these functions; viz.,  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$  and  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ . While it is possible in any particular case to derive the  $\lambda$ -development ( $\lambda = \lambda'$ ) of a real number by means of the defining equations and the interval system (15), §1 directly, yet a general description of such developments involve the partial and complete quotients in  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$  and  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ . That this description may refer ultimately to the defining equations and to the interval system (15), §1 alone, to these are referred the partial and complete quotients in question. Now in the first instance since  $\frac{1}{1-\lambda'}$  is a complete quotient in  $\lambda(\lambda')_{\lambda'}$  this is done by finding  $\lambda(\lambda')_{\lambda'}$  directly by means of the defining equations and the interval system (15), §1; in the second instance by showing how from  $\lambda(\lambda')_{\lambda'}$  to derive  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ . In the further consideration of the subject the partial and complete quotients entering in  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$  and  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$  will be considered as known.

In the types of continued fractions hitherto studied  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$  and  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$  do not play a conspicuous part. For  $\lambda' = 1$ , which is the case corresponding most closely to ordinary continued fractions  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'} = \infty$ ,  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = -1$ ; for  $\lambda' = \frac{1}{2}$ , which with very slight modification gives  $H_I$ ,  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'} = 2$ ,  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = -2$ ; for  $\lambda' = \frac{-1+\sqrt{5}}{2}$  i. e. for  $H_{II}$

$$\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'} = (3, 3, 3, \dots),$$

$$\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = (-2, -3, -3, \dots),$$

---

\* The idea that  $\lambda\left(\frac{1}{1-\lambda'}\right)_{\lambda'}$  and  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$  should first be derived and that their partial and complete quotients be considered as known, which effects a treatment of the general case, is due to Professor E. H. Moore.



where the real significance of these developments is veiled somewhat by their periodicity.

2.° Since  $0 < \lambda' \leq 1$ ,  $\lambda' \sim 1$  [§1, (15)]. When  $\lambda' = 1$ , the  $\lambda(\lambda')_{\lambda'} = 1$  and this case is disposed of completely. When  $\lambda' < 1$ , then  $\xi$  is determined by the equation

$$\lambda' = 1 - \frac{1}{\xi_1} = (1, \xi_1). \quad (1)$$

Hence

$$\lambda(\lambda')_{\lambda'} = (1, \xi_1), \quad (2)$$

where

$$\xi_1 = \frac{1}{1 - \lambda'} = 1 + \lambda' + \lambda'^2 + \dots > 1 + \lambda'. \quad (3)$$

When  $\xi_1$  is an integer  $\alpha_1$ , then  $\lambda(\lambda')_{\lambda'}$  cannot be extended further than in equation (2). When  $\xi_1$  is not an integer its corresponding integer  $\alpha_1$  is determined by reference to the interval system (15), §1. Then  $\xi_2$  is defined by the equation

$$\xi_1 = \alpha_1 - \frac{1}{\xi_2} = (\alpha_1, \xi_2) = \lambda(\xi_1)_{\lambda'}. \quad (4)$$

From the correspondence of  $\xi_1$  and  $\alpha_1$

$$\frac{1}{\xi_2} = -\overline{1 - \lambda'} \dots (\lambda'), \quad (5)$$

whence

$$\xi_2 = \frac{1}{1 - \lambda'} \dots \left(-\frac{1}{\lambda'}\right). \quad (6)$$

Now by equations (2) and (4)

$$\lambda(\lambda')_{\lambda'} = (1, \alpha_1, \xi_2). \quad (7)$$

When  $\xi_2$  is an integer  $\alpha_2$ , the process of development terminates with  $\xi_2$ . When  $\xi_2$  is not an integer then its corresponding integer is determined directly by reference to the interval system (15), §1.  $\xi_3$  is then defined by the equation

$$\xi_2 = \alpha_2 - \frac{1}{\xi_3}. \quad (8)$$

For  $\xi_2 > 0$ ,

$$\frac{1}{\xi_3} = -\overline{1 - \lambda'} \dots \lambda', \quad (9)$$

and

$$\xi_3 = \frac{1}{1 - \lambda'} \dots \left(-\frac{1}{\lambda'}\right); \quad (10)$$

for  $\xi_2 < 0$ ,

$$\frac{1}{\xi_3} = (-\lambda') \dots 1 - \lambda', \quad (11)$$

and

$$\xi_3 = \left(\frac{1}{\lambda'}\right) \dots - \frac{1}{1 - \lambda'}. \quad (12)$$

In the more general notation explained in §1, 3.° Equations (10) and (12) may be briefly written

$$\xi_3 = \text{sgn. } \xi_2 \left\{ \frac{1}{1 - \lambda'} \dots \left(-\frac{1}{\lambda'}\right) \right\}. \quad (13)$$

Combining equations (7) and (8)

$$\lambda(\lambda')_{\lambda'} = (1, \alpha_1, \alpha_2, \xi_3). \quad (14)$$

Again, when  $\xi_3$  is an integer the process of development necessarily terminates with  $\xi_3$ . When  $\xi_3$  is not an integer it may be replaced by

$$\alpha_3 - \frac{1}{\xi_4}$$

where  $\xi_3 \sim \alpha_3$  and

$$\xi_4 = \text{sgn. } \xi_3 \left\{ \frac{1}{1 - \lambda'} \dots \left(-\frac{1}{\lambda'}\right) \right\} = \text{sgn. } \alpha_3 \left\{ \frac{1}{1 - \lambda'} \dots \left(-\frac{1}{\lambda'}\right) \right\}. \quad (15)$$

Equation (14) now becomes

$$\lambda(\lambda')_{\lambda'} = (1, \alpha_1, \alpha_2, \alpha_3, \xi_4). \quad (16)$$

This process of successive reduction may be continued as long as the last complete quotient found is not an integer. The result stated generally is

$$\lambda(\lambda')_{\lambda'} = (1, \alpha_1, \alpha_2, \dots, \alpha_{i-1}, \xi_i), \quad (17)$$

when  $\xi_i \sim \alpha_i$ ,  $\xi_i - \alpha_i = -\frac{1}{\xi_{i+1}}$  and

$$\xi_{t+1} = \text{sgn. } \alpha_t \left\{ \frac{1}{1 - \lambda'} \dots \left(-\frac{1}{\lambda'}\right) \right\}, \quad t = 0, 1, \dots, i-1, \quad (18)$$

$i$  a positive integer equal to or greater than one.

By equation (18)

$$|\xi_{i+1}| > 1$$

and, hence,

$$|\alpha_{t+1}| \geq 1, \quad t = 0, 1, \dots, i-1.$$

When  $\xi_i$  for any value of  $i$ , say  $i = k$ , is identical with its corresponding integer, say  $\alpha_k$ , we shall uniformly write the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $\lambda'$  thus:

$$\lambda(\lambda')_{\lambda'} = (1, \alpha_1, \alpha_2, \dots, \alpha_k). \quad (19)$$



Applying the process just employed in the case of  $\lambda'$  to  $\xi_t$  we have

$$\lambda(\xi_t)_{\lambda'} = (\alpha_t, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), t = 0, 1, 2, \dots, i-1, \quad (20)$$

where, for  $t = 0$ ,  $\alpha_t = 1$ , and  $\xi_t = \lambda'$ .

3.° To derive  $\lambda(-\lambda')_{\lambda'}$  from  $\lambda(\lambda')_{\lambda'}$  we appeal to the symmetry of the interval system (15), §1 with respect to zero. We have only to change the sign of all elements entering equations (1)–(17) to make the investigation apply to the present case. Obviously, this involves no change in the form of equations (13) and (18). Hence

$$\lambda(-\lambda')_{\lambda'} = (-1, -\alpha_1, \dots, -\alpha_{i-1} - \xi_i), \quad (21)$$

where  $\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \xi_i$  have the values assigned in equation (17). In like manner it follows that

$$\lambda(-\xi_t)_{\lambda'} = (-\alpha_t, -\alpha_{t+1}, \dots, -\alpha_{i-1}, -\xi_i), t = 0, 1, \dots, i-1. \quad (22)$$

4.° When  $\xi_t > 0$ ,  $0 \leq t < i$ , and  $m$  a positive integer,  $m < \xi_t$ , then, since  $\xi_t - m$  corresponds to  $\alpha_t - m$

$$\lambda(\xi_t - m)_{\lambda'} = (\alpha_t - m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), \quad (23)$$

and for a like reason when  $m$  is any positive integer

$$\lambda(\xi_t + m)_{\lambda'} = (\alpha_t + m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i). \quad (24)$$

When  $\xi_t < 0$ ,  $0 \leq t < i$ , we have, corresponding to equations (23) and (24), respectively, the following equations:

$$\lambda(\xi_t + m)_{\lambda'} = (\alpha_t + m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), \quad (25)$$

$m$  an integer,  $0 \leq m < |\xi_t|$ ;

$$\lambda(\xi_t - m)_{\lambda'} = (\alpha_t - m, \alpha_{t+1}, \dots, \alpha_{i-1}, \xi_i), \quad (26)$$

$m$  any positive integer.

5.° To derive  $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$  from  $\lambda(\lambda')_{\lambda'}$  grant that

$$\begin{aligned} \lambda(\lambda')_{\lambda'} &= (1, 2_1, 2_2, \dots, 2_h, \xi_{h+1}) \\ &= (1, 2_1, 2_2, \dots, 2_h, \alpha_{h+1}, \dots, \alpha_{i-1}, \xi_i), \end{aligned} \quad (27)$$

where  $h$  is some positive integer and  $\alpha_{h+1} \neq 2$ .

By equation (27)

$$\lambda' = \frac{\xi_{h+1} - 1}{(h+1)\xi_{h+1} - h} \quad (28)$$

and

$$\frac{1}{\lambda'} = h + 1 + \frac{1}{\xi_{h+1} - 1}. \quad (29)$$

Equation (27) is a special case of the following: Let

$$x_0 = (a_0, \gamma_1, \gamma_2, \dots, \gamma_n, \xi_{n+1}),$$

where

$$\gamma_1 = \gamma_2 = \dots = \gamma_n \equiv \gamma.$$

Then

$$x_0 = a_0 - \frac{\phi_{n-1}(\gamma) \xi_{n+1} - \phi_{n-2}(\gamma)}{\phi_n(\gamma) \xi_{n+1} - \phi_{n-1}(\gamma)},$$

where

$$\phi_k(\gamma) = \sum_0^{\left[\frac{k}{2}\right]} (-1)^i \binom{k-i}{i} \gamma^{k-2i},$$

$$\phi_0(\gamma) = 1 \text{ and } \phi_{-1}(\gamma) = 0.$$

$$a) \quad \lambda'^2 + \lambda' - 1 > 0.$$

In this case

$$\lambda' > \frac{-1 + \sqrt{5}}{2} \equiv \tau, \quad (30)$$

and

$$\frac{1}{1-\lambda'} > 2 + \lambda'. \quad (31)$$

Consequently for the values of  $\lambda'$  now in question, the condition  $\alpha_{h+1} \neq 2$  is always fulfilled and under the hypothesis of equation (27)

$$\xi_{h+1} = \frac{1}{1-\lambda'} \dots \left(-\frac{1}{\lambda'}\right). \quad (32)$$

Also, it is true that

$$\xi_h > \frac{1}{1-\lambda} > 2 + \lambda, \quad (33)$$

and, consequently, that  $\alpha_h > 2$ , which is consistent with the hypothesis in equation (27) only when  $h = 0$ . Hence in this case

$$\xi_{h+1} = \frac{1}{1-\lambda'} > 0, \quad (34)$$

and, consequently,

$$h+1 < \frac{1}{\lambda'} \equiv h+1 + \frac{1}{\xi_1-1} < h+1 + \lambda'. \quad (35)$$

Therefore,

$$\frac{1}{\lambda'} \sim h+1, \quad (36)$$

and

$$\lambda \left( \frac{1}{\lambda'} \right)_{\lambda'} = (h+1, -(\xi_{h+1}-1)) = (h+1, -(\alpha_1-1), \\ -\alpha_2, \dots, -\alpha_{i-1}, -\xi_i), \tau < \lambda' < 1. \quad (37)$$



$$\text{b)} \quad 0 < \lambda' \leq \tau.$$

Bearing in mind the hypothesis with respect to  $\alpha_{h+1}$ , we have

$$\xi_{h+1} = 2 + \lambda' \dots \left(-\frac{1}{\lambda'}\right), \quad (38)$$

and consequently,

$$\frac{1}{\xi_{h+1} - 1} = \left(\frac{1}{1 + \lambda'}\right) \dots \frac{1}{1 + \lambda'}. \quad (39)$$

Hence

$$\begin{aligned} \frac{1}{\lambda'} &\equiv h + 1 + \frac{1}{\xi_{h+1} - 1} = \left(h + 1 + \frac{\lambda'}{1 + \lambda'}\right) \dots h' + 1 + \frac{1}{1 + \lambda'} \\ &= \left(h + \frac{1}{1 + \lambda'}\right) \dots h' + 1 + \frac{1}{1 + \lambda'}. \end{aligned} \quad (40)$$

Now, for  $\lambda' \leq \tau$ ,

$$h + \frac{1}{1 + \lambda'} \leq h + \lambda' \quad (41)$$

and

$$h + 1 + \lambda' \leq h + 1 + \frac{1}{1 + \lambda'} < h + 2. \quad (42)$$

Hence for the values of  $\lambda'$  under consideration  $\frac{1}{\lambda'}$  corresponds to  $h + 1$  or to  $h + 2$ .

$$\text{b}_1) \quad \frac{1}{\lambda'} \sim h + 1.$$

Then

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = (h + 1, -(\xi_{h+1} - 1)) \quad (43)$$

and

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = (h + 1, -(\alpha_{h+1} - 1), -\alpha_{h+2}, \dots, -\alpha_{i-1}, -\xi_i). \quad (44)$$

$$\text{b}_2) \quad \frac{1}{\lambda'} \sim h + 2.$$

In this case

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = \left(h + 2, \frac{\xi_{h+1} - 1}{\xi_{h+1} - 2}\right). \quad (45)$$

Since

$$h + 1 + \frac{1}{\xi_{h+1}} \leq h + 1 + \lambda', \quad (46)$$

then

$$0 < \xi_{h+1} \leq \frac{1}{\lambda'} + 1. \quad (47)$$

Now

$$\frac{\xi_{h+1} - 1}{\xi_{h+1} - 2} = \left(2_1, 2_2, \dots, 2_{k-2}, \frac{\xi_{h+1} - h + 1}{\xi_{h+1} - k}\right), \quad k \text{ an integer} \geq 2. \quad (48)$$

In order that

$$\frac{\xi_{h+1} - k + 1}{\xi_{h+1} - k} \equiv 1 + \frac{1}{\xi_{h+1} - k} \sim 2,$$

it is necessary and sufficient that

$$\frac{1}{\xi_{h+1} - k} = \lambda' \dots (1 + \lambda'), \quad (49)$$

or

$$\xi_{h+1} = \left( \frac{1}{1 + \lambda'} + k \right) \dots \frac{1}{\lambda'} + k. \quad (50)$$

By equation (47)  $\xi_{h+1}$  is less than the greater limit of this interval. Again, since

$$\xi_{h+1} = \alpha_{h+1} - 1 + \lambda', \quad (51)$$

then for  $0 < \lambda' < \tau$ , and for  $k < \alpha_{h+1} - 2$ ,  $\xi_{h+1}$  is certainly greater than the lesser interval limit in equation (50). Hence by equation (48)

$$\lambda \left( \frac{\xi_{h+1} - 1}{\xi_{h+1} - 2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\alpha_{h+1}-4}, 2_{\alpha_{h+1}-3}, \frac{\xi_{h+1} - \alpha_{h+1} + 2}{\xi_{h+1} - \alpha_{h+1} + 1} \right). \quad (52)$$

By the equation

$$\xi_{h+1} - \alpha_{h+1} = -\frac{1}{\xi_{h+1}}$$

equation (52) becomes

$$\lambda \left( \frac{\xi_{h+1} - 1}{\xi_{h+1} - 2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\alpha_{h+1}-3}, 2 + \frac{1}{\xi_{h+2} - 1} \right). \quad (53)$$

The further discussion of this equation presents two cases.

$$b_{2,1}) \quad 2 + \frac{1}{\xi_{h+2} - 1} \sim 2.$$

Equation (53) now becomes

$$\lambda \left( \frac{\xi_{h+1} - 1}{\xi_{h+1} - 2} \right)_{\lambda'} = (2_1, 2_2, \dots, 2_{\alpha_{h+1}-3}, 2_{\alpha_{h+1}-2}, -(\xi_{h+2} - 1)). \quad (54)$$

Combining this with equation (45)

$$\lambda \left( \frac{1}{\lambda'} \right)_{\lambda'} = (h+2, 2_1, 2_2, \dots, 2_{\alpha_{h+1}-3}, 2_{\alpha_{h+1}-2}, -(\alpha_{h+2} - 1), \dots, -\alpha_{i+1}, -\xi_i). \quad (55)$$

$$b_{2,2}) \quad 2 + \frac{1}{\xi_{h+1} - 1} \text{ does not correspond to } 2.$$

Then

$$\frac{1}{1 - \lambda'} < \xi_{h+1} < \frac{1}{\lambda'} + 1, \alpha_{h+2} > 2, \quad (56)$$



and the further reduction of the last complete quotient  $2 + \frac{1}{\xi_{h+1}-1}$  in equation (53) presents precisely the problem of equation (29) for  $0 < \lambda' \leq \tau$ , which has been partially solved under b), b<sub>1</sub>) and b<sub>2</sub>). It may be included in the following:

Let  $\alpha_{h+2} = \alpha_{h+3} = \dots = \alpha_{h+h_1+1} = 2 \neq \alpha_{h+h_1+2}$ ,  $h_1$  a positive integer. Then

$$2 + \frac{1}{\xi_{h+2}-1} = 3 + \frac{1}{\xi_{h+3}-1} = \dots = h_1 + 2 + \frac{1}{\xi_{h+h_1+2}-1} \quad (57)$$

In this case equation (53) reduces to

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\frac{a-3}{h+1}}, h_1 + 2 + \frac{1}{\xi_{h+h_1+2}-1} \right). \quad (58)$$

It may be remarked that for  $h_1 = 0$  this equation includes the case specified under equation (56).

Now, when

$$\xi_{h+h_1+2} - 1 = \left( \frac{1}{\lambda'} + 1 \right) \dots \left( -\frac{1}{\lambda'} \right), \quad (59)$$

then by equation (58)

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = (2_1, 2_2, \dots, 2_{\frac{a-3}{h+1}}, h + 2, -(\xi_{h+h_1+2} - 1)), \quad (60)$$

and the further extension of this development is immediate.

When, however,

$$\xi_{h+h_1+2} = \frac{1}{1-\lambda'} + 1 \dots \frac{1}{\lambda'} + 1, \quad (61)$$

it is readily shown that

$$h_1 + 2 + \frac{1}{\xi_{h+h_1+2}-1} \sim h_1 + 3. \quad (62)$$

In this case equation (58) becomes

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\frac{a-3}{h+1}}, 3 + h_1, \frac{\xi_{h+h_1+2}-1}{\xi_{h+h_1+2}-2} \right). \quad (63)$$

Applying to  $\frac{\xi_{h+h_1+2}-1}{\xi_{h+h_1+2}-2}$  the process employed in the reduction of  $\frac{\xi_{h+1}-1}{\xi_{h+1}-2}$  we have at once

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\frac{a-3}{h+1}}, 3 + h_1, 2_1, 2_2, \dots, 2_{\frac{a-3}{h+h_1+2}}, \frac{\xi_{h+h_1+2}-\alpha+2}{\xi_{h+h_1+2}-\alpha+1} \right). \quad (64)$$

By the equation

$$\xi_{h+h_1+2} - \alpha_{h+h_1+2} = -\frac{1}{\xi_{h+h_1+3}} \quad (65)$$

equation (64) reduces to

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+1}}, 3+h_1, 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+h_1+2}}, 2 + \frac{1}{\xi_{h+h_1+3}-1} \right). \quad (66)$$

Enough has been done to indicate the process by which the development in equation (66) may be carried to any desired extent. The result in general form may be written

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+1}}, 3+h_1, 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+h_1+2}}, 3+h_2, \dots, 3+h_{s-1}, 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+h_1+\dots+h_{s-1}+s-1}}, 2 + \frac{1}{\xi_{h+h_1+\dots+h_{s-1}+s}-1} \right), \quad (67)$$

where, as in equation (27)  $h$  is the number of successive plus twos immediately following the first element in  $\lambda(\lambda')_{\lambda'}$ ;  $h_1$  the number of successive plus twos immediately following  $\alpha_{h+1}$ ;  $h_2$  the number immediately following  $\alpha_{h+h_1+2}$ ;  $\dots$ ;  $h_s$  the number immediately following  $\alpha_{h+h_1+h_2+\dots+h_{s-1}+s}$ ; and where

$$\xi_t = \frac{1}{1-\lambda'} \dots \frac{1}{\lambda'} + 1, t = 1, 2, \dots, h+h_1+h_2+\dots+h_{s-1}+s. \quad (68)$$

When

$$\xi_{h+h_1+\dots+h_{s-1}+s+1} = \left( \frac{1}{\lambda'} + 1 \right) \dots \left( -\frac{1}{\lambda'} \right), \quad (69)$$

equation (67) may take the form

$$\lambda \left( \frac{\xi_{h+1}-1}{\xi_{h+1}-2} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+1}}, 3+h_1, 2_1, \dots, 2_{\frac{\alpha-3}{h+h_1+2}}, 3+h_2, \dots, 3+h_{s-1}, 2_1, 2_2, \dots, 2_{\frac{\alpha-3}{h+h_1+\dots+h_{s-1}+s-1}}, -(\xi_{h+h_1+\dots+h_{s-1}+s+1}-1) \right). \quad (70)$$

Returning now to equation (45) it may be written in the form

$$\lambda \left( \frac{1}{\lambda'} \right)_{\lambda'} = \left( h+2, 2_1, \dots, 2_{\frac{\alpha-3}{h+1}}, 3+h_1, 2_1, \dots, 2_{\frac{\alpha-3}{h+h_1+2}}, 3+h_2, \dots, 3+h_{s-1}, 2_1, \dots, 2_{\frac{\alpha-3}{h+h_1+\dots+h_{s-1}+s-1}}, 2 + \frac{1}{\xi_{h+h_1+\dots+h_{s-1}+s}-1} \right), \quad (71)$$

where condition (68) is satisfied. When in addition to this condition (70) is also satisfied, *i. e.*

$$\xi_{h+h_1+\dots+h_{s-1}+1} = \left(\frac{1}{\lambda'} + 1\right) \dots \left(-\frac{1}{\lambda'}\right), \quad (72)$$

then equation (71) takes the form

$$\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'} = \left(h+2, 2_1, 2_2, \dots, 2_{\frac{a-3}{h+1}}, 3+h_1, 2_1, \dots, 2_{\frac{a-3}{h+h_1+2}}, \right. \\ \left. 3+h_2, \dots, 3+h_{s-1}, 2_1, \dots, 2_{\frac{a-2}{h+h_1+\dots+h_{s-1}}}, -(\xi_{h+h_1+\dots+h_{s-1}+1} - 1)\right). \quad (73)$$

In equations (69)–(72)  $h, h_1, \dots, h_{s-1}$  preserve the meanings assigned under equation (67).

6.° Every case which may arise in the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $\frac{1}{\lambda'}$  has now been considered. The results are contained in equations (37), (44), (55), (60), (71), (73). From these equations may be drawn the following proposition.

THEOREM I.  $\lambda(\lambda')_{\lambda'}$  and  $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$  from elements of certain ranks on differ in and only in the signs of their elements; or in  $\lambda(\lambda')_{\lambda'}$   $\xi_t = \frac{1}{1-\lambda'} \dots \frac{1}{\lambda'} + 1$ ,  $t = 1, 2, \dots, n-1$ ,  $0 < \lambda' < \tau$  and  $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$  is derived from  $\lambda(\lambda')_{\lambda'}$  by the rule embodied in equation (71).

7.° As a convenient notation not only for this section but also for the following let us write

$$\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = (\beta_1, \beta_2, \dots, \beta_{i-1}, \eta_i), \quad (74)$$

$i$  a positive integer not zero.

Let  $\eta_t$  denote the complete quotient corresponding to  $\beta_t$ ,  $t = 0, 1, 2, i-1$ . When  $\eta_t$  is an integer let equation (74) be uniformly written

$$\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'} = (\beta_1, \beta_2, \dots, \beta_h). \quad (75)$$

Since  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$ , because of the symmetry of the interval system with respect to zero, is derived from  $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$  simply by changing the signs of all the elements in the latter it follows at once by Theorem I that



THEOREM II.  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$  and  $\lambda(\lambda')_{\lambda'}$  are ultimately alike or in  $\lambda(\lambda')_{\lambda'} \xi_{t+1} < 0$ ,  
 $|\xi_{t+1}| = \frac{1}{1-\lambda'} \cdots \frac{1}{\lambda'} + 1$ ,  $t = 0, 1, \dots, n-1$ ,  $0 < \lambda' \leq \tau$ , and the elements in  
 $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$  differ from those in equation (71) only in sign.

In theorems I. and II. note that  $n$  may be taken as great as we please, or, if not, that the element in  $\lambda(\lambda')_{\lambda'}$  following  $\alpha_k$  is  $\infty$ .

8.° a)  $\lambda(1-\lambda')_{\lambda'}$ .

There are two cases for consideration.

a<sub>1</sub>)  $1-\lambda' < \lambda'$ .

In this case  $1-\lambda' \sim 0$  and

$$\begin{aligned} \lambda(1-\lambda')_{\lambda'} &= 0 - \frac{1}{1-\lambda'} = \left(0, -\frac{1}{1-\lambda'}\right) = (0, -\xi_1) \\ &= (0, -\alpha_1, -\alpha_2, \dots, -\alpha_{n-1}, -\xi_n). \end{aligned} \quad (76)$$

a<sub>2</sub>)  $1-\lambda' \geq \lambda'$ .

Here  $1-\lambda' \sim 1$  and

$$\lambda(1-\lambda')_{\lambda'} = \left(1, \frac{1}{\lambda'}\right) = (1, -\beta_1, -\beta_2, \dots, -\beta_{n-1}, -\eta_n). \quad (77)$$

b)  $\lambda\left(\frac{\lambda'}{\lambda'-1}\right)_{\lambda'}$ .

In this case

$$\frac{\lambda'}{\lambda'-1} = -\left\{\frac{1}{1-\lambda'} - 1\right\} = -(\xi_1 - 1), \quad (78)$$

and since  $\xi_1 > 1$  formula (23) applies. Hence

$$\lambda\left(\frac{\lambda'}{\lambda'-1}\right)_{\lambda'} = (-(\alpha_1 - 1), -\alpha_2, \dots, -\alpha_{n-1}, -\xi_n). \quad (79)$$

c)  $\lambda\left(\frac{\lambda'-1}{\lambda'}\right)_{\lambda'}$ .

In this case

$$\frac{\lambda'-1}{\lambda'} = -\left(\frac{1}{\lambda'} - 1\right) \quad (80)$$

and, since  $\frac{1}{\lambda'} \geq 1$  formula (23) again applies. Hence

$$\lambda\left(\frac{\lambda'-1}{\lambda'}\right)_{\lambda'} = (-(\beta_1 - 1), -\beta_2, \dots, -\beta_{n-1}, -\eta_n). \quad (81)$$

9.° It will be convenient here to complete the treatment of the case partially dealt with under 4°; viz.,  $\lambda(\xi_t - m)_{\lambda'}$ ,  $m$  any integer. Because of the symmetry of the interval system we need consider in detail only the case  $m > \xi_t > 0$ . There are three cases:

$$\text{a) } \xi_t - \alpha_t = -1 \dots -\lambda'.$$

Now, since  $\xi_t - \alpha_t \sim -1$

$$\xi_t - m \equiv \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t + 1) \quad (82)$$

Hence

$$\begin{aligned} \lambda(\xi_t - m)_{\lambda'} &= \left( -(m - \alpha_t + 1) - \frac{1}{\frac{1}{\xi_t - \alpha_t + 1}} \right) \\ &= \left( -(m - \alpha_t + 1), -\frac{1}{\xi_t - \alpha_t + 1} \right). \end{aligned} \quad (83)$$

Since

$$\xi_t - \alpha_t = -\frac{1}{\xi_{t+1}}$$

equation (83) becomes

$$\lambda(\xi_t - m)_{\lambda'} = \left( -(m - \alpha_t + 1), 1 + \frac{1}{\xi_{t+1} - 1} \right). \quad (84)$$

The last complete quotient comes under the case presented in equation (29), so that we may say that  $\lambda(\xi_t - m)_{\lambda'}$  and  $\lambda\left(\frac{1}{\lambda'}\right)_{\lambda'}$  are in the case under consideration ultimately alike.

$$\text{b) } \xi_t - \alpha_t = (-\lambda') \dots 0.$$

In this case, since  $\xi_t - \alpha_t \sim 0$ , and  $\xi_t - \alpha_t < 0$

$$\xi_t - m \equiv \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t). \quad (85)$$

Hence

$$\lambda(\xi_t - m)_{\lambda'} = -(m - \alpha_t) - \frac{1}{\frac{1}{\xi_t - \alpha_t}} = \left( -(m - \alpha_t), -\frac{1}{\xi_t - \alpha_t} \right). \quad (86)$$

Since

$$\xi_t - \alpha_t = -\frac{1}{\xi_{t+1}}$$

equation (86) becomes

$$\lambda(\xi_t - m)_{\lambda'} = \left( -(m - \alpha_t), \alpha_{t+1}, \dots, \alpha_{n-1}, \xi_n \right). \quad (87)$$

$$c) \quad \xi_t - \alpha_t = 0 \dots (\lambda').$$

This case is most conveniently treated under two sub-cases  $c_1)$  and  $c_2)$ .

$$c_1) \quad \xi_t - \alpha_t - 1 \sim -1.$$

Now

$$\xi_t - m \equiv \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t). \quad (88)$$

Hence

$$\begin{aligned} \lambda(\xi_t - m)_{\lambda'} &= \left( -(m - \alpha_t), -\frac{1}{\xi_t - \alpha_t} \right) = \left( -(m - \alpha_t), \xi_{t+1} \right) \\ &= \left( -(m - \alpha_t), \alpha_{t+1}, \alpha_{t+2}, \dots, \alpha_{n-1}, \xi_n \right). \end{aligned} \quad (89)$$

$$c_2) \quad \xi_t - \alpha_t - 1 \sim 0.$$

In this case

$$\xi_t - m = \xi_t - \alpha_t - (m - \alpha_t) \sim -(m - \alpha_t - 1). \quad (90)$$

Therefore

$$\lambda(\xi_t - m)_{\lambda'} = \left( -(m - \alpha_t - 1), -\frac{1}{\xi_t - \alpha_t - 1} \right) = \left( -(m - \alpha_t - 1), \frac{\xi_{t+1}}{\xi_{t+1} + 1} \right). \quad (91)$$

It follows from the correspondence in equation (90) that

$$\left| \frac{\xi_{t+1}}{\xi_{t+1} + 1} \right| > 1. \quad (92)$$

Hence  $\xi_{t+1} < 0$ . Now writing equation (91) in the form

$$\lambda(\xi_t - m) = \left( -(m - \alpha_t - 1), 1 + \frac{1}{-\xi_{t+1} - 1} \right) \quad (93)$$

it is evident that its further reduction comes under the case considered in equation (29), where  $\xi_{h+1}$  is to be replaced by  $-\xi_{t+1}$ .

As already pointed out  $\lambda(\xi_t - m)_{\lambda'}$ ,  $\xi_t < 0$ ,  $m > |\xi_t|$  can be derived from the cases just treated by considerations of symmetry. This, then completes the case partially treated under 4.<sup>o</sup> Accordingly we shall hereafter consider  $\lambda(\xi_t + m)$ ,  $t$  any positive integer,  $m$  any integer, as known.

### § 3. $\lambda$ -developments ( $\lambda = \lambda'$ , $0 < \lambda' \leq 1$ ).

1.<sup>o</sup> The object of this section is to derive the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $x_0$ ,  $x_0$  any real number not zero, to discuss certain properties of this development, and by means of these to set up a norm whereby to decide when a given terminating, or finite, development is a  $\lambda$ -development ( $\lambda = \lambda'$ ). In a finite  $\lambda$ -development all the elements except possibly the last are integers. The last element may be integral, fractional or irrational, but except when otherwise specified, as in the last section following theorems I. and II., is in all cases finite.

2.<sup>o</sup> The finite  $\lambda$ -development ( $\lambda = \lambda'$ ) of a real number  $x_0$ , ( $x_0 \neq 0$ ), is



derived from the equations arising when to  $i$  the values 0 to  $n-1$  are successively assigned in the equation

$$x_i = a_i - \frac{1}{x_{i+1}}, \quad x_i \sim a_i, a_i \text{ an integer}, \quad (1)$$

by the elimination of  $x_1, x_2, \dots, x_{n-1}$ . The result in the usual notation is

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n). \quad (2)$$

To examine more closely the values of  $a_0, a_1, \dots, a_{n-1}, x_n$  it is necessary to make explicit and precise the correspondence of  $x_i$  and  $a_i$ .

When the correspondence of  $x_i$  and  $a_i$  may be completely determined by the unmodified interval system (15), §1, always possible for  $i=0$ , then

$$x_i - a_i = \text{sgn. } x_i \{ (-\lambda') \dots (\lambda') \}, |x_i| < \lambda', 0 \leq i < n; \quad (3)$$

$$x_i - a_i = \text{sgn. } x_i \{ -\overline{1 - \lambda'} \dots (\lambda') \}, |x_i| \geq \lambda', 0 \leq i < n. \quad (4)$$

Since

$$x_i - a_i = -\frac{1}{x_{i+1}}$$

equations (3) and (4), respectively, give

$$x_{i+1} = \text{sgn. } x_i \left\{ \left( -\frac{1}{\lambda'} \right) \dots \left( -\frac{1}{\lambda'} \right) \right\}, |x_i| < \lambda', 0 \leq i < n; \quad (5)$$

$$x_{i+1} = \text{sgn. } x_i \left\{ \frac{1}{1 - \lambda'} \dots \left( -\frac{1}{\lambda'} \right) \right\}, |x_i| \geq \lambda', 0 \leq i < n. \quad (6)$$

The intervals in equations (5) and (6) are composed of normal unit intervals and two special or end intervals. Each end interval has one limit number coincident with a limit of the interval under consideration in equation (5) or (6). The end intervals may in certain cases considered later be identical. In the discussion of equations (5) and (6) two cases present themselves according as normal unit intervals or end intervals are concerned. It will be convenient to consider first the case involving only normal intervals.

3.° *Normal intervals.*—When  $|x_0| \geq \lambda'$  and all complete quotients  $x_i, i = 1, 2, \dots, n$ , lie in normal unit intervals, then equations (4) and (6) are satisfied for  $i = 0, 1, 2, \dots, n-1$ . When  $|x_0| < \lambda'$  and all complete quotients  $x_i, i = 1, 2, \dots, n-1$ , lie in normal unit intervals then equations (3) and (5) are satisfied for  $i = 0$ , and equations (4) and (6) for  $i = 1, 2, \dots, n-1$ . Since for  $|x_i| < \lambda'$   $\text{sgn. } x_{i+1} = -\text{sgn. } x_i$ , then every  $x_i, a_i, x_{i+1}$  satisfying equations (3) and (5) satisfy also equations (4) and (6) in the sense that  $x_i, a_i, x_{i+1}$  will be

found in their respective intervals as assigned by equations (4) and (6). Adopting the convention that for  $|x_i| < \lambda' \operatorname{sgn}. a_i = \operatorname{sgn}. x_i$ , an equation necessarily true in all other cases, equations (4) and (6) may take the form

$$x_i - a_i = \operatorname{sgn}. a_i \{ - \overline{1 - \lambda'} \dots (\lambda') \}, i = 0, 1, 2, \dots, n-1, \quad (7)$$

$$x_{i+1} = \operatorname{sgn}. a_i \left\{ \frac{1}{1 - \lambda'} \dots \left( - \frac{1}{\lambda'} \right) \right\}, i = 0, 1, 2, \dots, n-1, \quad (8)$$

where  $x_i$  and  $x_{i+1}$  lie in normal unit intervals.

4.° *The general case.*—To dispose of this case we begin by establishing the following equation:

$$x_{i+1} = \operatorname{sgn}. a_f \{ Y_{i,r,f} \dots Z_{i,s,f} \}, i = f, f+1, \dots, n-1, \quad (9)$$

where  $r$  is the number of successive elements belonging to the left end interval and immediately preceding  $x_{i+1}$ ,  $s$  the number of successive elements belonging to the right end interval and immediately preceding  $x_{i+1}$ ,  $x_f$  corresponds to  $a_f$  and lies in a normal unit interval, and the left and the right end interval limits are to be interpreted by the following rules: for the left end interval limit

$$Y_{i,h,f} = \left\{ \frac{\xi_{h+1}}{(-\eta_{h+1})} \right\}^* \text{ when } \operatorname{sgn}. a_f \operatorname{sgn}. a_{i-h} = \left\{ \frac{+1}{-1} \right\}; \quad (10)$$

for the right end interval limit

$$Z_{i,h,f} = \left\{ \frac{(\eta_{h+1})}{-\xi_{h+1}} \right\} \text{ when } \operatorname{sgn}. a_f \operatorname{sgn}. a_{i-h} = \left\{ \frac{+1}{-1} \right\}, 0 \leq h < i. \quad (11)$$

It should be remarked that  $r$  and  $s$  are simultaneously different from zero when and only when the left and the right end intervals are identical.

Let equation (9) be satisfied for  $i = i_1$ , then it is to be shown that it is satisfied when  $i = i_1 + 1$ . For  $i = i_1$ , equation (9) becomes

$$x_{i_1+1} = \operatorname{sgn}. a_f \{ Y_{i_1,r_1,f} \dots Z_{i_1,s_1,f} \}, \quad (12)$$

where  $r = r_1$   $s = s_1$  for  $i = i_1$ .

a) Let  $a_{i_1+1}$  fall in the left end interval and not in the right.

Then, to work out the equations for  $\operatorname{sgn}. a_f = +1$ ,

$$x_{i_1+1} - a_{i_1+1} = \xi_{r_1+1} - a_{i_1+1} \dots (\lambda'), \quad a_{i_1+1} > 0; \quad (13)$$

$$x_{i_1+1} - a_{i_1+1} = (-\eta_{r_1+1} - a_{i_1+1}) \dots (1 - \lambda'), \quad a_{i_1+1} < 0. \quad (14)$$

---

\*  $\xi_{h+1}$  and  $\eta_{h+1}$  have the meanings assigned in §2.

The corresponding equations for  $x_{i_1+2}$  are, accordingly,

$$x_{i_1+2} = \xi_{r_1+2} \dots \left(-\frac{1}{\lambda'}\right), \text{ since } \xi_{r_1+1} \sim a_{i_1+1}, a_{i_1+1} > 0; \quad (15)$$

$$x_{i_1+2} = (-\eta_{r_1+2}) \dots -\frac{1}{1-\lambda'}, \text{ since } -\eta_{r_1+1} \sim a_{i_1+1}, a_{i_1+1} < 0. \quad (16)$$

In view of the symmetry of the interval system the complete result may be at once written

$$x_{i_1+2} = \text{sgn. } a_f \{ Y_{i_1+1, r_1+1, f} \dots Z_{i_1+1, 0, f} \}. \quad (17)$$

This is identical with equation (9) for  $i = i_1 + 1, r = r_1 + 1, s = 0$ .

b) Let  $x_{i_1+1}$  fall in the right end interval and not in the left. Then

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \{ -1 + \lambda' \dots Z_{i_1, s_1, f} - a_{i_1+1} \text{sgn. } a_f \}, a_{i_1+1} > 0; \quad (18)$$

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \{ (\lambda') \dots Z_{i_1, s_1, f} - a_{i_1+1} \text{sgn. } a_f \}, a_{i_1+1} < 0. \quad (19)$$

Since  $Z_{i_1, s_1, f} \sim a_{i_1+1} \text{sgn. } a_f$  we have from equations (18) and (19)

$$x_{i_1+2} = \text{sgn. } a_f \{ Y_{i_1+1, 0, f} \dots Z_{i_1+1, s_1+1, f} \}, \quad (20)$$

an equation identical with equation (9) for  $i = i_1 + 1, r = 0, s = s_1 + 1$ .

c)  $x_{i_1+1}$  lies at the same time in both end intervals. Then

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \{ Y_{i_1, r_1, f} - a_{i_1+1} \text{sgn. } a_f \dots Z_{i_1, s_1, f} \dots - a_{i_1+1} \text{sgn. } a_f \} \quad (21)$$

and

$$x_{i_1+2} = \text{sgn. } a_f \{ Y_{i_1+1, r_1+1, f} \dots Z_{i_1+1, s_1+1, f} \}, \quad (22)$$

since

$$Y_{i_1, r_1, f} - a_{i_1+1} \text{sgn. } a_f = -\frac{1}{Y_{i_1+1, r_1+1, f}} \quad (23)$$

and

$$Z_{i_1, s_1, f} - a_{i_1+1} \text{sgn. } a_f = -\frac{1}{Z_{i_1+1, s_1+1, f}}. \quad (24)$$

Equation (22) is identical with equation (9) for  $i = i_1 + 1, r = r_1 + 1, s = s_1 + 1$ .

d)  $x_{i_1+1}$  falls in neither end interval but in a normal unit interval. Then the correspondence of  $x_{i_1+1}$  and  $a_{i_1+1}$  is completely determined by the interval system (15), §1. Hence (cf. equation (7))

$$x_{i_1+1} - a_{i_1+1} = \text{sgn. } a_f \{ -\overline{1 - \lambda'} \dots (\lambda') \} \quad (25)$$

and

$$x_{i_1+2} = \text{sgn. } a_f \{ Y_{i_1+1, 0, f} \dots Z_{i_1+1, 0, f} \}, \quad (26)$$

The last equation is identical with equation (9) for  $i = i_1 + 1, r = s = 0$ .

Now it has been shown that if equation (9) holds for  $i = i_1$  then it is true for  $i = i_1 + 1, f \leq i \leq n - 1$ . For  $i = f$ , equation (9) is identical with equation (8) and therefore a valid equation. Hence equation (9) is true also for  $i = f + 1, f + 2, \dots, n - 1$ .



$x_f$  lies in a normal interval certainly for  $f=0$  and consequently equation (9) defines completely every complete quotient  $x_1, x_2, \dots, x_n$  in  $\lambda(x_0)_{\lambda'}$  in terms of preceding representative integers and the complete quotients in  $\lambda(\lambda')_{\lambda'}$  and in  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}^*$

When in equation (13)  $a_{i+1} = \alpha_k$ , or, in equation (14)  $a_{i+1} = -\beta_l$ , then

$$x_{i+2} = \infty \dots \left(-\frac{1}{\lambda'}\right), \quad (27)$$

$$x_{i+2} = (-\infty) \dots -\frac{1}{1-\lambda'}. \quad (28)$$

Equations (27) and (28) may be derived directly from equations (15) and (16) by the convention  $\xi_{k+1} = \eta_{l+1} = \infty$ .† A like remark applies to equations (17)–(22), inclusive.

Let  $Y_{i,h,f}$  correspond to  $A_{i,h,f}$  and  $Z_{i,h,f}$  to  $B_{i,h,f}$ . Then corresponding to equations (10) and (11) we have

$$A_{i,h,f} = \left\{ \begin{array}{c} \alpha_{h+1} \\ -\beta_{h+1} \end{array} \right\} \text{ when } \text{sgn. } a_f \text{sgn. } a_{i-h} = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\}; \quad (29)$$

and

$$B_{i,h,f} = \left\{ \begin{array}{c} \beta_{h+1} \\ -\alpha_{h+1} \end{array} \right\} \text{ when } \text{sgn. } a_f \text{sgn. } a_{i-h} = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\}, \quad 0 \leq h < i, \quad (30)$$

where  $r$  and  $s$  have the meanings assigned under equation (9) and  $x_f$ , corresponding to  $a_f$ , lies in a normal unit interval.

Hence, corresponding to equation (9) we may write

$$a_{i+1} = \text{sgn. } a_f \{A_{i,r,f} \dots B_{i,s,f}\}, \quad (31)$$

Equations (9) and (31) completely solve our first problem; viz., that of assigning to every real number  $x_0$  a  $\lambda$ -development ( $\lambda = \lambda'$ ). They are the fundamental equations of our theory, including equation (8) as a particular case.

The preceding results may be resumed in the following theorem:

**THEOREM I.** *For  $x_0$  a real non-zero number*

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n), \quad n > 0,$$

\*And herein, by way of remark, appears the fundamental importance of  $\lambda(\lambda')_{\lambda'}$  and  $\lambda\left(-\frac{1}{\lambda'}\right)_{\lambda'}$  in every  $\lambda$ -development ( $\lambda = \lambda'$ ).

† Cf. §1, 2°; also remark under theorem II, §2.

where

- 1.°  $x_0 \sim a_0$ ,
- 2.°  $a_{i+1} = \text{sgn. } a_0 \{A_{i, r, 0} \dots B_{i, s, 0}\}, i = 0, 1, 2, \dots, > n - 2$ ,
- 3.°  $x_n = \text{sgn. } a_0 \{Y_{i, r_1, 0} \dots Z_{i, s_1, 0}\}.$

the expressions in the interval limits being interpreted by equations (10), (11), (29) and (30), with the addition that when  $i$  takes the particular value  $n$ ,  $r$  and  $s$  take respectively the particular values  $r_1$  and  $s_1$ .

5.° Employing the notation of theorem I we state the converse proposition:

THEOREM II. Conversely, when for a continued fraction

$$x_0 = (a_0, a_1, \dots, a_{n-1}, x_n), \quad n > 0, a_i, i = 0, 1, 2, \dots, n - 1, \text{ an integer,}$$

the following conditions are satisfied:

- 1.°  $a_{i+1} = \text{sgn. } a_0 \{A_{i, r, 0} \dots B_{i, s, 0}\}, i = 0, 1, 2, \dots, n - 2.$
- 2.°  $x_n = \text{sgn. } a_0 \{Y_{i, r_1, 0} \dots Z_{i, s_1, 0}\}$

then

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n).$$

The proof is as follows:

$$\text{a) } n = 1.$$

Condition 2° asserts the correspondence of  $a_0 - \frac{1}{x_1}$  and  $a_0$ . Therefore, since

$$x_0 = a_0 - \frac{1}{x_1}. \quad \lambda(x_0)_{\lambda'} = (a_0, x_1). \quad (32)$$

$$\text{b) } n > 1.$$

The interval assigned to  $x_{n-1}$  in accordance with theorem II. is

$$x_{n-1} = \text{sgn. } a_0 \{Y_{n-2, r_2, 0} \dots Z_{n-2, s_2, 0}\}, \quad (33)$$

where  $r_2$  and  $s_2$  are the values of  $r$  and  $s$  for  $i = n - 1$ .

By condition 1° for  $i = n - 2$ ,

$$a_{n-1} = \text{sgn. } a_0 \{A_{n-2, r_2, 0} \dots B_{n-2, s_2, 0}\} \quad (34)$$

and by condition 2° for  $i = n$

$$-\frac{1}{x_n} = \text{sgn. } a_0 \left\{ -\frac{1}{Y_{n-1, r_1, 0}} \dots -\frac{1}{Z_{n-1, s_1, 0}} \right\}. \quad (35)$$

$$\text{b}_1) \quad r_1 \neq 0, \quad s_1 \neq 0.$$

Therefore,  $r_1 = r_2 + 1, s_1 = s_2 + 1$ .

Hence

$$\begin{aligned} a_{n-1} - \frac{1}{x_n} &= \text{sgn. } a_0 \left\{ A_{n-2, r_2, 0} - \frac{1}{Y_{n-1, r_1, 0}} \dots B_{n-2, s_2, 0} - \frac{1}{Z_{n-1, s_1, 0}} \right\} \\ &= \text{sgn. } a_0 \{Y_{n-2, r_2, 0} \dots Z_{n-2, s_2, 0}\}, \end{aligned} \quad (36)$$

this last reduction being made by means of the correspondence of  $Y_{n-2, r_2, 0}$  and  $A_{n-2, r_2, 0}$  on the one hand and of  $Z_{n-2, s_2, 0}$  and  $B_{n-2, s_2, 0}$  on the other. Since  $x_{n-1} = a_{n-1} - \frac{1}{x_n}$ ,  $x_{n-1}$  lies in the interval required by equation (33).

$$b_2) \quad r_1 > 0, s_1 = 0.$$

In this case  $r_1 = r_{2+1}$  and  $a_{n-1}$  lies in the left end interval, equation (34), and not in the right. Under this hypothesis

$$-\frac{1}{x_n} = \text{sgn. } a_0 \left\{ -\frac{1}{Y_{n-1, r_1, 0}} \dots - \frac{1}{Z_{n-1, 0, 0}} \right\}, \quad (37)$$

and, consequently

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \left\{ A_{n-2, r_2, 0} - \frac{1}{Y_{n-1, r_1, 0}} \dots A_{n-2, r_2, 0} - \frac{1}{Z_{n-1, 0, 0}} \right\}, \quad (38)$$

where  $A_{n-2, r_2, 0}$  is to be interpreted by equation (29).

The left interval limit in equation (38) reduces at once to the left interval limit in equation (33). We proceed to show by evaluating the right interval limit that the interval in equation (38) always lies in or at most coincides with the interval in equation (33). Since  $a_{n-1}$  lies in the left end interval of equation (33) and not in the right, the right limit number must certainly lie without the normal unit interval associated with  $a_{n-1}$ . We now show that the interval in equation (38) contains at most one normal unit interval.

For the right interval limit we have

$$\begin{aligned} (\alpha_{r_2+1} + \lambda'), & \quad \text{when } a_{n-1} = \alpha_{r_2+1} \text{sgn. } a_0 \text{ and } \text{sgn. } a_0 = \text{sgn. } a_{n-2-r} = \text{sgn. } a_{n-1}; \\ \alpha_{r_2+1} + 1 - \lambda', & \quad \text{when } a_{n-1} = \alpha_{r_2+1} \text{sgn. } a_0 \text{ and } \text{sgn. } a_0 = \text{sgn. } a_{n-2-r_2} = -\text{sgn. } a_{n-1}; \\ (\beta_{r_2+1} + \lambda'), & \quad \text{when } a_{n-1} = \beta_{r_2+1} \text{sgn. } a_0 \text{ and } \text{sgn. } a_0 = -\text{sgn. } a_{n-2-r_2} = \text{sgn. } a_{n-1}; \\ \beta_{r_2+1} + 1 - \lambda', & \quad \text{when } a_{n-1} = \beta_{r_2+1} \text{sgn. } a_0 \text{ and } \text{sgn. } a_0 = -\text{sgn. } a_{n-2-r_2} \\ & \quad \quad \quad = -\text{sgn. } a_{n-1}; \end{aligned}$$

In every instance the right interval limit in equation (33) coincides with one limit of the normal unit interval represented by  $a_{n-1}$ , while the left interval limit, as already pointed out, is identical with the left interval limit of equation (33), which always lies in the normal unit interval associated with  $a_{n-1}$ , or coincides with a limit number of this interval. Hence the interval in equation (38) is at most coextensive with a normal unit interval and is included in the interval in equation (33). Hence in this case  $a_{n-1} - \frac{1}{x_n}$  ( $\equiv x_{n-1}$ ) is in the required interval.



$$b_3) \quad r_1 = 0, \quad s_1 > 0.$$

Here  $s_1 = s_2 + 1$  and  $a_{n-1}$  lies in the right end interval and not in the left. This case differs so slightly from the preceding that it is presented in merest outline. The interval equation for  $a_{n-1} - \frac{1}{x_n}$  corresponding to equation (38) will have its right limit number identical with the right limit number in equation (33). The left limit numbers save as to order are identical with the right limit numbers found in the preceding case. Hence the conclusion in this case as in the preceding that  $a_{n-1} - \frac{1}{x_n} (\equiv x_{n-1})$  lies in the interval of equation (33).

$$b_4) \quad r_1 = s_1 = 0.$$

In this case  $a_{n-1}$  lies in a normal unit interval, i. e., the interval assigned to  $x_{n-1}$  by equation (33) contains in this case certainly one normal unit interval; viz., that associated with  $a_{n-1}$ . It is to be shown that  $a_{n-1} - \frac{1}{x_n}$  lies in this normal unit interval.

For this case

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \left\{ a_{n-1} \text{sgn. } a_0 - \frac{1}{Y_{n-1,0,0}} \dots a_{n-1} \text{sgn. } a_0 - \frac{1}{Z_{n-1,0,0}} \right\}. \quad (39)$$

When  $\text{sgn. } a_0 \text{sgn. } a_{n-1} = +1$ , equation (39) becomes

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \{ \overline{a_{n-1} - 1 + \lambda'} \text{sgn. } a_0 \dots (a_{n-1} + \lambda') \text{sgn. } a_0 \}; \quad (40)$$

when  $\text{sgn. } a_0 \text{sgn. } a_{n-1} = -1$ , equation (39) becomes

$$a_{n-1} - \frac{1}{x_n} = \text{sgn. } a_0 \{ (a_{n-1} - \lambda') \text{sgn. } a_0 \dots \overline{a_{n-1} + 1 - \lambda'} \text{sgn. } a_0 \}. \quad (41)$$

Hence by equations (40) and (41)  $a_{n-1} - \frac{1}{x_n} (\equiv x_{n-1})$  lies in the normal unit interval associated with  $a_{n-1}$ , and hence, also, in the interval of equation (33).

Having considered the several cases of our theorem, which may arise under b), we may now conclude that

$$x_0 = (a_0, a_1, \dots, a_{n-2}, x_{n-1}). \quad x_{n-1} = a_{n-1} - \frac{1}{x_n},$$

satisfies the conditions

$$\begin{aligned} 1.^{\circ} \quad & a_{i+1} = \text{sgn. } a_0 \{ A_{i,r,0} \dots B_{i,s,0} \}, \quad i = 0, 1, 2, \dots, n-3, \\ 2.^{\circ} \quad & x_{n-1} = \text{sgn. } a_0 \{ Y_{i,r,0} \dots Z_{i,s,0} \}. \end{aligned}$$

This reduction may be repeatedly applied until we have finally

$$x_0 = (a_0, x_1),$$

a case considered under a). The result of this investigation is that  $x_i$  and  $a_i$ ,  $i = 0, 1, 2, \dots, n-1$ , correspond, when  $x_i$  is calculated by the equation

$$x_i = a_i - \frac{1}{x_{i+1}}.$$

This correspondence proves that the continued fraction in question is the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $x_0$  and, therefore, establishes theorem II.

This completes the second problem proposed; namely, to establish a criterion whereby to decide if a given finite continued fraction be a  $\lambda$ -development ( $\lambda = \lambda'$ ).

6.° A few properties of  $\lambda$ -developments ( $\lambda = \lambda'$ ) implicit in the preceding formulas are here made explicit for the purposes of the further discussion of our theory.

Since the end intervals are at most coextensive with normal unit intervals then when  $x_i \sim a_i$  and lies on the zero side of  $a_i$

$$|x_i - a_i| \leq 1 - \lambda'; \quad (42)$$

and when  $x_i$  lies on the infinity side of  $a_i$

$$|x_i - a_i| < \lambda' \quad i = 0, 1, \dots, n-1. \quad (43)$$

Hence, for the corresponding cases,

$$|x_{i+1}| \geq \frac{1}{1 - \lambda'} = \xi_1, \quad (44)$$

$$|x_{i+1}| > \frac{1}{\lambda'} = |\eta_0|. \quad (45)$$

Now when  $x_i$  lies on the  $\left\{ \begin{smallmatrix} \text{zero} \\ \text{infinity} \end{smallmatrix} \right\}$  side of  $a_i$ ,  $\text{sgn. } x_{i+1} = \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} \text{sgn. } x_i$ , and, conversely. Evidently, also, when  $x_i$  lies on the  $\left\{ \begin{smallmatrix} \text{zero} \\ \text{infinity} \end{smallmatrix} \right\}$  side of  $a_i$ ,  $\text{sgn. } a_{i+1} = \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} \text{sgn. } a_i$ , and conversely. Further (cf. §2, equation (3))

$$\xi_1 > 1 + \lambda' \quad (46)$$

and, hence

$$\alpha_1 \geq 2.$$

Therefore,

THEOREM III. When  $\text{sgn. } a_{i+1} = \text{sgn. } a_i$ ,  $|a_{i+1}| \geq 2$ ,  $i = 0, 1, 2, \dots, n-1$ .

Since

$$|\eta_0| = \frac{1}{\lambda'} \geq 1,$$

then

THEOREM IV. When  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$ ,  $|a_{i+1}| \geq 1$ ,  $i = 0, 1, 2, \dots, n-1$ .

And, consequently

THEOREM V. When  $|a_{i+1}| = 1$ ,  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$ ,  $i = 0, 1, \dots, n-1$ .  
When

$$\lambda' \leq -\frac{1 + \sqrt{5}}{2} \equiv \tau,$$

then

$$\frac{1}{\lambda'} \geq 1 + \lambda';$$

and, when

$$\lambda' > \tau,$$

then

$$\frac{1}{\lambda'} < 1 + \lambda'.$$

In consequence theorem V may be stated thus:

THEOREM VI. When  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$  then for  $\lambda' > \tau_1$ ,  $|a_{i+1}| \geq 1$  and for  $\lambda' \leq \tau_1$ ,  $|a_{i+1}| \geq 2$ .

When in equation (17), §2,

$$\alpha_1 = \alpha_2 = \dots = \alpha_j = 2,$$

then

$$\xi_{j+1} = \frac{1 - j\lambda'}{1 - (j+1)\lambda'}.$$

Now for

$$a_{f+1} = \alpha_1, \quad a_{f+2} = \alpha_2, \dots, \quad a_{f+j} = \alpha_j$$

equation (9) becomes

$$x_{i+1} = \text{sgn. } a_f \{ Y_{i,j,f} \dots Z_{i,j,f} \}, \quad i = f+j, \quad (47)$$

or

$$x_{i+1} = \text{sgn. } a_f \left\{ \frac{1 - j\lambda'}{1 - (j+1)\lambda'} \dots \left( -\frac{1}{\lambda'} \right) \right\}. \quad (48)$$

When  $j < \frac{1}{\lambda'} \leq j+1$ ,  $\text{sgn. } x_{i+1} = -\text{sgn. } a_i$ . Hence

THEOREM VII. The maximum number of  $\text{sgn. } a_f \cdot 2$ 's immediately preceded by  $a_f$ , the representative integer of a normal unit interval, and followed by an element with the same sign as  $a_f$ , which may enter a  $\lambda$ -development ( $\lambda = \lambda'$ ) is  $j-1$  where

$$j < \frac{1}{\lambda'} \leq j+1.$$

When the convention is adopted that  $\text{sgn. } \infty = \text{sgn. } a_f$ , then the maximum number is  $j$ .

When

$$\frac{1}{\lambda'} = j+1,$$

then

$$\lambda(\lambda')_N = (1, 2_1, 2_2, \dots, 2_j).$$



In this case we may say

THEOREM VIII. *When in  $\lambda(x_0)_{\lambda'}$ ,  $\frac{1}{\lambda'} = j + 1$  there enters a maximum number of successive 2's having the sign of  $a_j$  then they are immediately followed by an element with the opposite sign or by  $\infty$ .*

For those values of  $\lambda'$  which permit  $+2$  to enter a  $\lambda$ -development ( $\lambda = \lambda'$ ) preceded by a positive element the sequence

$$(a_0, 2_1, 2_2, \dots, 2_{j-2}, s, 2_1, 2_2, \dots, 2_{j-2}, s, 2_1, 2_2, \dots, 2_{j-2}, s, x_n) \quad (49)$$

is a  $\lambda$ -development ( $\lambda = \lambda'$ ), where  $a_0 > 0$ ,  $s = a_2 + 1$ ,  $a_2$  the minimum positive integer which may follow  $j - 2$  successive plus twos without requiring an immediate change of sign,  $j$  is defined by the inequality

$$j < \frac{1}{\lambda'} \leq j + 1,$$

and

$$x_n = \xi_1 \dots (\eta_1).$$

The truth of this statement is evident from theorem II.

#### §4. *Convergents of $\lambda$ -developments ( $\lambda = \lambda'$ ).*

1.° When  $p_{i-1}$ ,  $p_i$ ,  $p_{i+1}$ , and  $q_{i-1}$ ,  $q_i$ ,  $q_{i+1}$  satisfy the equations

$$p_{i+1} = a_{i+1} p_i - p_{i-1}, \quad p_{-1} = 1, \quad p_0 = a_0, \quad (1)$$

$$q_{i+1} = a_{i+1} q_i - q_{i-1}, \quad q_{-1} = 0, \quad q_0 = 1, \quad i = 0, 1, 2, \dots, n-2. \quad (2)$$

where  $a_{i+1}$  is an element in the continued fraction

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n), \quad (3)$$

then it may be shown that

$$\frac{p_i}{q_i}, \quad i = 0, 1, 2, \dots, n-1, \quad (4)$$

is the  $(r+1)^{\text{th}}$  convergent of the continued fraction (3).

Directly by equation (1) follows

LEMMA I. *When  $|p_i| > |p_{i-1}|$ , then  $\text{sgn. } p_{i+1} = \text{sgn. } a_{i+1} \cdot \text{sgn. } p_i$ ,  $i = 0, 1, \dots, n-1$ .*

And, again,

LEMMA II. *When  $|p_{i+1}| > |p_i| > |p_{i-1}|$ , then  $|p_{i+2}| > |p_{i+1}|$ ,  $i = 0, 1, \dots, n-3$ .*

For, when  $|a_{i+2}| > 2$ , the conclusion of lemma II follows at once from the hypothesis by equation (1). When  $|a_{i+2}| = 1$ , then  $\text{sgn. } a_{i+2} = -\text{sgn. } a_{i+1}$  and, lemma I,

$$|p_{i+2}| = |p_{i+1}| + |p_i| > |p_{i+1}|.$$

Now

$$p_0 = a_0, p_1 = a_1 p_0 - 1, p_2 = a_2 p_1 - p_0, \dots, \quad (5)$$

and in all cases

$$|p_2| \geq |p_1| \geq |p_0|. \quad (6)$$

Hence, by virtue of lemma II,

LEMMA III.

$$|p_{i+1}| \geq |p_i|, \quad i = 0, 1, 2, \dots, n-2.$$

The inequality sign prevails in all cases in the lemma except as indicated below. When  $\lambda(x_0)_{\lambda'}$  or  $\lambda(-x_0)_{\lambda'}$  takes the form

$$(0, a_1, 1, \dots) \quad a_1 < 0 \quad (7)$$

the equality sign is taken for  $i = 0, 1, 2, \dots, h$ , or when  $\lambda(x_0)_{\lambda'}$  or  $\lambda(-x_0)_{\lambda'}$  takes the form

$$(1, 2_1, \dots), \quad (8)$$

the equality sign is taken for  $i = 0, 1, 2, \dots, h$ , where  $h$  is the number of successive plus "twos" following immediately the element one in the continued

fractions one (7) and (8).  $h$  does not exceed  $\left\{ j - \frac{1}{\lambda'} \right\}$  where

$$\left\{ \begin{array}{l} j < \frac{1}{\lambda'} < j+1 \\ j+1 = \frac{1}{\lambda'} \end{array} \right\}^* \quad (9)$$

By precisely similar reasoning on equation (2) we have corresponding to lemmas I, II., and III.

LEMMA IV. When  $|q_{i+1}| > |q_i|$ , then  $\text{sgn. } q_{i+1} = \text{sgn. } a_{i+1} \text{sgn. } q_i$ ,  $i = 0, 1, 2, \dots, n-2$ .

LEMMA V. When  $|q_{i+1}| \geq |q_i| \geq |q_{i-1}|$ , then  $|q_{i+2}| > |q_{i+1}|$ ,  $i = 1, 2, \dots, n-3$ .

LEMMA VI.  $|q_{i+1}| \geq |q_i|$ ,  $i = 0, 1, 2, \dots, n-2$ .

In lemma VI the equality sign is to be taken for  $i = 0, 1$  when  $\lambda(x_0)_{\lambda'}$  or  $\lambda(-x_0)_{\lambda'}$  assumes the form

$$(0, -1, 1, \dots). \quad (10)$$

2.° As in the theory of simple continued fractions it can be shown that

$$x_0 = \frac{p_i x_{i+1} - p_{i-1}}{q_i x_{i+1} - q_{i-1}}, \quad 0 \leq i \leq n-1, \quad (11)$$

where  $x_0$  is given by equation (3). By equation (11) after a slight reduction

$$x_0 - \frac{p_i}{q_i} = \frac{-1}{q_i^2 \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right)}. \quad (12)$$

\* Cf. Theorem VII, §3.

When  $\lambda(x_0)_{\lambda'}$  terminates with a complete quotient  $x_n = a_n$ ,  $x_{n+1}$  is to be taken as infinite. In this instance, since  $\frac{q_{i-1}}{q_i}$  is finite, equation (12) gives

$$\text{Hence} \quad x_0 - \frac{p_n}{q_n} = 0. \quad (13)$$

THEOREM I. *The value of a terminating  $\lambda$ -development ( $\lambda = \lambda'$ ) may be expressed by a rational fraction.*

3.° When  $\lambda(x_0)_{\lambda'}$  does not terminate with  $x_n = a_n$  however great  $n$  may be, there are two cases to

$$\text{a) } \lambda' = (0) \dots (1).$$

When  $\text{sgn. } a_{i+1} = \text{sgn. } a_1$ ,  $|x_{i+1}| > \frac{1}{1 - \lambda'} > 1 + \lambda'$ . And, since

$$\text{then by equation (12)} \quad \frac{q_{i-1}}{q_i} < 1,$$

$$\left| x_0 - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2 \lambda'}, \quad i = 0, 1, 2, \dots, n-1. \quad (14)$$

When  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$ ,  $|x_{i+1}| > \frac{1}{\lambda'}$ . Let  $\lambda' = 1 - \varepsilon$ . Then

$$\frac{1}{\lambda'} > 1 + \varepsilon.$$

Hence in this case by equation (12)

$$\left| x_0 - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2 \varepsilon}, \quad 0 \leq i \leq n-1. \quad (15)$$

$$\text{b) } \lambda' = 1.$$

In this case  $\text{sgn. } x_{i+1} = -\text{sgn. } a_i \equiv -\text{sgn. } \frac{q_{i-1}}{q_i}$  (lemma IV.) and consequently

$$\left| x_{i+1} - \frac{q_{i-1}}{q_i} \right| = |x_{i+1}| + \left| \frac{q_{i-1}}{q_i} \right| > 1, \quad 0 \leq i \leq n-1. \quad (16)$$

$$\text{Hence} \quad \left| x_0 - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2}, \quad 0 \leq i \leq n-1. \quad (17)$$

Now since  $|q_i|$  increases without limit with  $i$ , an immediate inference from lemma IV. and following remark,  $i$  may be chosen so great that the second members of the inequalities (14), (15) and (17) become less than any given non-zero positive number however small. Therefore in all cases

$$\lim_{i \rightarrow \infty} \frac{p_i}{q_i} = x_0. \quad (18)$$



4.° Consider now the non-terminating continued fraction

$$(a_0, a_1, \dots, a_i, a_{i+1}, \dots, a_{i+n}, \dots), \quad (19)$$

whose first  $i + n + 1$  elements however great  $i$  and  $n$  may be satisfy for  $\lambda = \lambda'$  the condition

$$a_{j+1} = \text{sgn. } a_0 \{ A_{j,r,0} \dots B_{j,s,0} \}, j = 0, 1, \dots, i + n, \quad (20)$$

where  $r$  and  $s$  are defined under equation (9) and  $A_{j,r,0}$ ,  $B_{j,s,0}$  under equations (29) and (30) of §3. Consider, also, the associated sequence of successive convergents derived from the  $p$ 's and  $q$ 's arising out of equation (19); viz.,

$$\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}, \dots, \frac{p_{i+n}}{q_{i+n}}, \dots \quad (21)$$

This sequence, as will be shown, always has a limit. This limit will be taken by definition for the value of the non-terminating continued fraction (19). The  $\lambda$ -development ( $\lambda = \lambda'$ ) of the value so defined of this infinite  $\lambda$ -development ( $\lambda = \lambda'$ ) is, with exceptions to be specified, identically so far as partial quotients are concerned, the continued fraction (19).

It can be shown that

$$\frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i} = - \left\{ \frac{1}{q_i q_{i+1}} + \frac{1}{q_{i+1} q_{i+2}} + \dots + \frac{1}{q_{i+n-2} q_{i+n-1}} + \frac{1}{q_{i+n-1} q_{i+n}} \right\}; \quad (22)$$

and, consequently, that

$$\left| \frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i} \right| \leq \frac{1}{|q_i| \cdot |q_{i+1}|} + \frac{1}{|q_{i+1}| \cdot |q_{i+2}|} + \dots + \frac{1}{|q_{i+n-2}| \cdot |q_{i+n-1}|} + \frac{1}{|q_{i+n-1}| \cdot |q_{i+n}|}. \quad (23)$$

Hence, since for  $i > 2$  the  $q$ 's increase in absolute value, for  $i > 2$

$$\left| \frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i} \right| \leq \frac{1}{N(N+1)} + \frac{1}{(N+1)(N+2)} + \dots + \frac{1}{(N+n-2)(N+n-1)} + \frac{1}{(N+n-1)(N+n)} \leq \frac{1}{N} - \frac{1}{N+n}, \quad (24)$$

where  $N = |q_i|$ . Now  $i$  may be chosen so great that for all values of  $n$ ,

$$\frac{1}{N} - \frac{1}{N+n} < \epsilon$$

where  $\varepsilon$  is an arbitrarily assigned non-zero positive number. In truth when  $i$  is so chosen that  $\frac{1}{N} < \varepsilon$ , then for all values of  $n$

$$\left| \frac{p_{i+n}}{q_{i+n}} - \frac{p_i}{q_i} \right| < \varepsilon.$$

Hence the sequence (21) has a limit. Let us call this limit  $x_0$ . Then by definition

$$x_0 = (a_0, a_1, \dots, a_i, a_{i+1}, \dots, a_{i+n}, \dots). \quad (25)$$

5.° It will now be shown, with exceptions to be noted that

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_i, a_{i+1}, \dots, a_{i+n}, x_{i+n+1}) \quad (26)$$

for all values of  $i$  and  $n$  however great.

By analogy to the definition in equation (25)  $x_j$  is defined by the equation

$$x_j = (a_j, a_{j+1}, \dots, a_{i+n}, \dots), \quad j = 0, 1, 2, \dots, i+n, \dots. \quad (27)$$

Let  $\frac{P_i}{Q_i}$  be defined by the equation

$$\frac{P_i}{Q_i} = (a_1, a_2, \dots, a_i). \quad (28)$$

Now when  $a_i$  satisfies the relation

$$a_i = \text{sgn. } a_0 \{ A_{i-1, r, 0} \dots B_{i-1, s, 0} \}, \quad (\text{theorem II, §3}), \quad (29)$$

then

$$(a_0, a_1, \dots, a_i) \quad (30)$$

is the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $\frac{p_i}{q_i}$ . Hence

$$\frac{p_i}{q_i} = a_0 - \frac{1}{\frac{P_i}{Q_i}}, \quad (31)$$

where

$$\frac{p_i}{q_i} \sim a_0.$$

a) Let it now be granted that there is no integer  $n_1$  such that for all values of  $i$ ,  $i > n_1$ , sequence (30) is not the  $\lambda \left( \frac{p_i}{q_i} \right)_{\lambda'}$ . Then taking limits of both members of equation (31) for  $i = \infty$ , we have

$$x_0 = a_0 - \frac{1}{x_1}, \quad (32)$$

where, because of the correspondence of  $\frac{p_i}{q_i}$  and  $a_0$  for certain values of  $i$  under the conditions imposed,

$$x_0 - a_0 = \text{sgn. } a_0 \{ -1 + \lambda' \dots \lambda' \}. \quad (33)$$

When

$$x_0 = a_0 + \lambda' \operatorname{sgn}. a_0, \quad (34)$$

$x_0$  does not correspond to  $a_0$ ; in all other cases  $x_0 \sim a_0$ .

When  $x_0 \sim a_0$ , then it may be shown that either  $x_1 \sim a_1$  or  $x_1 = a_1 + \lambda' \operatorname{sgn}. a_1$ . And, again, when  $x_1 \sim a_1$  then  $x_2 \sim a_2$ , or  $x_2 = a_2 + \lambda' \operatorname{sgn}. a_2$ , etc.

Hence equation (26) is true or else for some value of  $j$

$$x_j = a_j + \lambda' \operatorname{sgn}. a_j \quad 0 \leq j \leq i + n. \quad (35)$$

b) Let it now be supposed that there is an integer  $n_1$  such that for all values of  $i$ ,  $i > n$ , sequence (30) is not  $\lambda \left( \frac{p_i}{q_i} \right)$ ; In this case select a number  $v_i$  such that the sequence

$$(a_0, a_1, \dots, a_i + v_i) \quad (36)$$

is a  $\lambda$ -sequence ( $\lambda = \lambda'$ ).  $v_i$  may at the same time satisfy the inequalities

$$0 < |v_i| \leq 1.$$

Now let  $\frac{p'_i}{q'_i}$  be the  $(i + 1)^{\text{th}}$  convergent of the sequence (36). Then

$$\frac{p'_i}{q'_i} - \frac{p_i}{q_i} = \frac{p_{i-1}(a_i + v_i) - p_{i-2}}{q_{i-1}(a_i + v_i) - q_{i-1}} - \frac{p_i}{q_i} = \frac{p_i + p_{i-1}v_i}{q_i + q_{i-1}v_i} - \frac{p_i}{q_i}. \quad (37)$$

Hence

$$\frac{p'_i}{q'_i} - \frac{p_i}{q_i} = \frac{v_i}{q_i(q_i + q_{i+1}v_i)}. \quad (38)$$

Passing to limits for  $i = \infty$ , we have

$$\lim_{i=\infty} \frac{p'_i}{q'_i} = \lim_{i=\infty} \frac{p_i}{q_i} = x_0. \quad (39)$$

Now by theorem II, §3,  $\frac{p'_i}{q'_i} \sim a_0$ . Hence

$$x_0 - a_0 = \operatorname{sgn}. a_0 \{ -1 + \lambda' \dots \lambda' \} \quad (40)$$

and either  $x_0 \sim a_0$ , or

$$x_0 = a_0 + \lambda' \operatorname{sgn}. a_0. \quad (41)$$

When  $x_0 \sim a_0$ , then it may be shown as in the case of  $x_0$  either that  $x_1 \sim a_1$  or

$$x_1 = a_1 + \lambda' \operatorname{sgn}. a_1.$$

When  $x_1 \sim a_1$ , then, again, it may be shown that either  $x_2 \sim a_2$ , or

$$x_2 = a_2 + \lambda' \operatorname{sgn}. a_2.$$



By continuing this reasoning the conclusion is reached that equation (26) is true or that for some value of  $j$

$$x_j = a_j + \lambda' \operatorname{sgn}. a_j, \quad 0 \leq j \leq i + n. \quad (42)$$

Equations (35) and (42) are formally identical. Evidently

$$\lambda(x_j)_{\lambda'} = (a_j + \operatorname{sgn}. a_j, \alpha_1 \operatorname{sgn}. a_j, \alpha_2 \operatorname{sgn}. a_j, \alpha_3 \operatorname{sgn}. a_j, \dots). \quad (43)$$

Therefore, when equation (26) is not true

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{j-1}, a_j + \operatorname{sgn}. a_j, \alpha_1 \operatorname{sgn}. a_j, \alpha_2 \operatorname{sgn}. a_j, \alpha_3 \operatorname{sgn}. a_j, \dots), \quad 0 \leq j \leq i + n. \quad (44)$$

It has now been shown that  $\lambda(x_0)_{\lambda'}$  is identical with sequence (19) for as many successive partial quotients beginning with the first, as we please or that it takes the particular form (44).

As an example of an exception consider the infinite continued fraction

$$(2, -3, -3_2, -3_3, \dots), \quad (45)$$

in which  $-3_j, j = 0, 1, 2, \dots, n, n$  finite, satisfies equation (20) for

$$\lambda' = \tau \equiv \frac{-1 + \sqrt{5}}{2}.$$

For  $n = \infty$ , however, by definition we have from sequence (45)

$$x_0 = 2 + \tau, \quad (46)$$

whence

$$\lambda(x_0)_\tau = (3, 3, 3, 3, 3, \dots). \quad (47)$$

5.° A method of approximate calculation should provide a ready means of finding a sequence of numbers approaching a given number, of determining the error made by taking one of these numbers for the given number, and, finally, of making the approximation as close as may be desired.

When in the decimal approximation to a given number, a certain number of significant figures are taken, then a number of significant figures greater by one, and then, greater by two, etc., in the sequence of numbers thus formed each is a closer approximation than the preceding and the error committed by taking any term of this sequence for the given number is less than one unit of the order of the last significant figure to the right in the term chosen. Finally this error may be made as small as may be desired simply by making the number of significant figures in the approximate value sufficiently great.

To derive the corresponding properties of  $\lambda$ -developments ( $\lambda = \lambda'$ ) we start from the equation

$$x_0 - \frac{p_i}{q_i} = \frac{-1}{q_i^2 \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right)}. \quad (48)$$

But

$$x_0 - \frac{p_{i+1}}{q_{i+1}} = x_0 - \frac{p_i}{q_i} - \left( \frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} \right) = x_0 - \frac{p_i}{q_i} + \frac{1}{q_i q_{i+1}}. \quad (49)$$

Now by equation (48), since also

$$q_{i+1} = a_{i+1} q_i - q_{i-1}, \quad (50)$$

we have

$$\begin{aligned} x_0 - \frac{p_{i+1}}{q_{i+1}} &= \frac{-1}{q_i^2 \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right)} + \frac{1}{q_i^2 \left( a_{i+1} - \frac{q_{i-1}}{q_i} \right)} \\ &= \frac{x_{i+1} - a_{i+1}}{q_i^2 \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right) \left( a_{i+1} - \frac{q_{i-1}}{q_i} \right)}. \end{aligned} \quad (51)$$

Since

$$x_{i+1} - a_{i+1} = -\frac{1}{x_{i+2}},$$

the last equation may take the form

$$x_0 - \frac{p_{i+1}}{q_{i+1}} = \frac{1}{q_i^2 \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right) \left( a_{i+1} - \frac{q_{i-1}}{q_i} \right)} \cdot \frac{-1}{x_{i+2}}. \quad (52)$$

Now

$$\text{sgn.} \left( a_{i+1} - \frac{q_{i-1}}{q_i} \right) = \text{sgn.} \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right)$$

and, consequently, the first factor in the second member of equation (52) is positive. Hence by replacing  $i+1$  by  $i$ , equation (52) may be written in the form

$$x_0 - \frac{p_i}{q_i} = \frac{-A_i}{x_{i+1}}, \quad (53)$$

where  $A_i$  is a positive number. Therefore

**THEOREM II.** When  $x_{i+1} < 0$ ,  $x_0 > \frac{p_i}{q_i}$  and when  $x_{i+1} > 0$ ,  $x_0 < \frac{p_i}{q_i}$ .

It is now to be shown that

$$\left| x_0 - \frac{p_i}{q_i} \right| > \left| x_0 - \frac{p_{i+1}}{q_{i+1}} \right|. \quad (54)$$

By equation (48)

$$x_0 - \frac{p_{i+1}}{q_{i+1}} = \frac{-1}{q_{i+1}^2 \left( x_{i+1} - \frac{q_i}{q_{i+1}} \right)}. \quad (55)$$

Now, on the other hand,

$$\begin{aligned} q_i^2 \left( x_{i+1} - \frac{q_{i-1}}{q_i} \right) &= q_i^2 \left( a_{i+1} - \frac{1}{x_{i+2}} - \frac{q_{i-1}}{q_i} \right) = q_i \left( a_{i+1} q_i - q_{i-1} - \frac{q_i}{x_{i+2}} \right) \\ &= \frac{q_i q_{i+1} \left( x_{i+2} - \frac{q_i}{q_{i+1}} \right)}{x_{i+2}}. \end{aligned} \quad (56)$$

Hence

$$x_0 - \frac{p_i}{q_i} = \frac{-x_{i+2}}{q_i q_{i+1} \left( x_{i+2} - \frac{q_i}{q_{i+1}} \right)}. \quad (57)$$

Since  $|x_{i+2}| > |q_i|$  and  $|q_{i+1}| > |q_i|$

$$\left| \frac{-x_{i+2}}{q_i q_{i+1} \left( x_{i+2} - \frac{q_i}{q_{i+1}} \right)} \right| > \left| \frac{-1}{q_{i+1}^2 \left( x_{i+2} - \frac{q_i}{q_{i+1}} \right)} \right|. \quad (58)$$

By virtue of equation (55) this establishes inequality (54). Hence

THEOREM III. *In the continued fraction*

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n)$$

$\frac{p_{i+1}}{q_{i+1}}$  is a closer approximation to  $x_0$  than is  $\frac{p_i}{q_i}$ ,  $0 \leq i < n$ .

In  $\lambda(x_0)_{\lambda'}$  we shall get a relatively close approximation to  $x_0$  by stopping with the element  $a_{i+1}$  when both  $x_{i+1}$  and  $x_{i+2}$  are relatively great numerically with respect to  $a_i$ .

By equation (48) the error made in replacing  $x_0$  by a convergent of a definite order may be determined; by inequality (54) it follows that the higher the order of the convergent the closer the approximation to  $x_0$ ; and from the convergency of  $\lambda$ -developments ( $\lambda = \lambda'$ ) that the approximation to  $x_0$  may be made as close as desired by a convergent of a sufficiently high order.  $\lambda$ -developments ( $\lambda = \lambda$ ) are, therefore, well adapted to approximate calculations.

### §5. *Reverse $\lambda$ -Developments.*

1.° The treatment of certain questions in the theory of binary quadratic forms and, in particular, those concerning  $\lambda$ -developments ( $\lambda = \lambda'$ ) representing the square roots of integers not perfect squares is simplified when based on a system of continued fractions permitting reversal. The property of reversal belongs to simple continued fractions, to  $H_I$  and with a slight qualification to  $H_{II}$ . It is proposed to examine to what extent this property prevails in the system here under consideration. In particular it is desirable to specify what, if any,



values of  $\lambda'$  exist such that for every  $\lambda$ -development ( $\lambda = \lambda'$ ) there is a reverse  $\lambda$ -development for at least one value of  $\lambda$ , say  $\lambda''$ , independent of the particular  $\lambda$ -development in question. It will be found that the  $\lambda$ -developments ( $\lambda = \lambda'$ ,  $\lambda' = 1$ ), added to  $H_I$  and  $H_{II}$  completes the list of such developments.

2.° Equation (2), §4, may be written in the form

$$\frac{q_{i+1}}{q_i} = a_{i+1} - \frac{q_{i-1}}{q_i}, \quad i = 0, 1, \dots, n-2. \quad (1)$$

Adopting the notation  $Q_i = \frac{q_i}{q_{i-1}}$ , equation (1) becomes

$$Q_{i+1} = a_{i+1} - \frac{1}{Q_i}, \quad i = 0, 1, 2, \dots, n-1. \quad (2)$$

Let  $i$  take for the moment a fixed value  $n_1$ . Then equation (2) becomes

$$Q_{n_1+1} = a_{n_1+1} - \frac{1}{Q_{n_1}}, \quad 0 \leq n_1 < n-1. \quad (3)$$

From the equations derived from equation (2) letting  $i$  take successively the values  $0, 1, 2, \dots, n_1$  eliminate  $Q_2, Q_3, \dots, Q_{n_1}$ . The result may be expressed thus:

$$Q_{n_1+1} = (a_{n_1+1}, a_{n_1}, \dots, a_2, a_1). \quad (4)$$

The second member of this equation is called a reverse development to the  $\lambda$ -development ( $\lambda = \lambda'$ ) of  $x_0$ , where

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n), \quad n > n_1 + 1, \quad (5)$$

or, simply, a reverse development. When the second member of equation (4) satisfies the conditions under theorem II, §3, for  $\lambda = \lambda''$ , it will be referred to as a reverse  $\lambda$ -development ( $\lambda = \lambda''$ ).

3.° The necessary and sufficient condition that the second member of equation (4) be a  $\lambda$ -development ( $\lambda = \lambda''$ ) is, of course given by theorem II, §3; but for our present purpose it will be convenient to derive in another form a necessary condition.

When  $\text{sgn. } a_{i+1} = \text{sgn. } a_i$ , then by equation (2)

$$|Q_{i+1}| < |a_{i+1}|;$$

and, when  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$ ,

$$|Q_{i+1}| > |a_{i+1}|, \quad 0 < i \leq n_1.$$

When  $i = 0$ ,  $Q_1 = a_1$ .

Denote the maximum absolute value of  $\frac{1}{Q_{i+1}}$ ,  $0 < i \leq n_1$ , when  $\text{sgn. } a_{i+1} = \text{sgn. } a_i$  by  $g_{\lambda'}$ , and the like maximum of  $\frac{1}{Q_{i+1}}$ ,  $0 < i \leq n_1$ , when  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$  by  $g'_{\lambda'}$ . Now since the extent of interval associated with any non-zero integer is at most unity and further since all normal unit intervals have the same extent on the zero side of their representative integers and, again, the same extent on the side toward infinity, in order that there exist a value of  $\lambda$ , say  $\lambda'_1$ , for which the second member of equation (4) is a  $\lambda$ -development it is necessary that

$$g_{\lambda'} + g'_{\lambda'} \leq 1. \quad (6)$$

This necessary condition being satisfied, does there exist a value of  $\lambda_1$ , say  $\lambda''$ , for which  $\lambda(x_0)_{\lambda'}$  has a reverse  $\lambda$ -development ( $\lambda = \lambda''$ ) for every value of  $x_0$ ? The necessary criterion by which to answer this question may be derived from equation (6). Let  $x_0$  run through the whole range of real values and for every such value of  $x_0$  from the corresponding  $g_{\lambda'}$  and  $g'_{\lambda'}$ . Let  $\gamma_{\lambda'}$  denote the maximum value or the maximum limit of the  $g_{\lambda'}^s$ , and  $\gamma'_{\lambda'}$ , the maximum value or the maximum limit of the  $g_{\lambda'}'^s$ .

Then the necessary condition for the existence of a value of  $\lambda_1$ , say  $\lambda''$ , for which for every  $x_0$   $\lambda(x_0)_{\lambda'}$  has a reverse  $\lambda$ -development ( $\lambda = \lambda''$ ) is

$$\gamma_{\lambda'} + \gamma'_{\lambda'} \leq 1. \quad (7)$$

4.° The values of  $g_{\lambda'}$  and  $g'_{\lambda'}$  are to be derived from the minimum values or minimum limits of  $|Q_i|$ ,  $i = 1, 2, \dots, n_1 - 1$ , in the particular  $\lambda(x_0)_{\lambda'}$ , under consideration.

Replacing in equation (2)  $i$  by  $i - 1$  we have

$$Q_i = a_i - \frac{1}{Q_{i-1}}. \quad (8)$$

Under the hypothesis that  $\text{sgn. } a_i = \text{sgn. } a_{i-1}$ ,  $|Q_i|$  is a minimum when and only when  $|a_i|$  and  $|Q_{i-1}|$  take the minimum values consistent with the laws of  $\lambda$ -developments ( $\lambda = \lambda'$ ). The step by step application of this reasoning leads to the conclusion that  $|Q_i|$  is a minimum when  $a_1, a_2, \dots, a_{i-1}, a_i$  all agree in sign and have the least absolute values permissible in  $\lambda$ -developments ( $\lambda = \lambda'$ ). When  $|a_i| \geq |a_{i-1}| + 1$ , then  $|Q_i| \geq |Q_{i-1}|$ ,  $1 \leq i \leq n_1$ .

When  $\text{sgn. } a_i = -\text{sgn. } a_{i-1}$ ,  $|Q_i|$  is a minimum if  $|a_i|$  is a minimum and  $|Q_{i-1}|$  a maximum.

To apply test (6) note that

$$(a_0, 2, a_2, \dots, a_k, 2, a_2, \dots, a_k, 2, a_n), \quad n = 2k + 2, \quad (10)$$

where all the elements except  $a_n$  are positive is formally a  $\lambda$ -development ( $\lambda = \lambda', 0 < \lambda' < \frac{1}{2}$ ). In this case

$$Q_i = (2, a_k, \dots, a_2, 2), \quad i = k + 1, \quad (11)$$

and

$$g_{\lambda'} \geq \frac{1}{2 - \frac{1}{a_k}}. \quad (12)$$

Again

$$Q_i = (2, a_k, \dots, a_2, 2, a_k, \dots, a_2, 2), \quad i = 2k + 1. \quad (13)$$

Since by hypothesis  $a_n < 0$ ,

$$g'_{\lambda'} \geq \frac{1}{2 - \frac{1}{a_k}}. \quad (14)$$

Therefore

$$g_{\lambda'} + g'_{\lambda'} \geq \frac{2}{2 - \frac{1}{a_k}} > 1, \quad (15)$$

and the necessary condition ( $b_0$ ) is not satisfied. Hence for every value of  $\lambda'$ ,  $0 < \lambda' < \frac{1}{2}$ , there exists values of  $x_0$  such that for  $\lambda(x_0)_{\lambda'}$  there is no value of  $\lambda$  for all positive values of  $n_1$  and  $n$ ,

$$Q_{n_1} = (a_{n_1}, a_{n_1-1}, \dots, a_2, a_1), \quad 0 < n_1 < n, \quad (16)$$

is a reverse  $\lambda$ -development.

For  $\frac{1}{2} < \lambda' < \tau$ ,  $\tau \equiv \frac{-1 + \sqrt{5}}{2}$  consider the sequence

$$(a_0, \dots, a_j, 2, a_{j+2}, 3_1, 3_2, \dots, 3_k, 2, a_n), \quad n = j + k + 4, \quad (17)$$

where  $a_0 > 0$ ,  $a_j < 0$ ,  $a_{j+2} > 0$ ,  $a_n < 0$ , and properly chosen.

When choice has been made of a particular value for  $\lambda'$  then certainly the  $a$ 's must fulfill certain conditions but which in no case involve a change of sign of the  $a$ 's. To determine approximately  $g_{\lambda'}$  let

$$Q_i = (2, a_i, \dots, a_1), \quad i = j + 1.$$

Then, certainly,

$$2 + \frac{1}{|a_j| + 1} \leq |Q_i| \leq 2 + \frac{1}{|a_j| - 1}, \quad (18)$$

and, correspondingly,

$$\frac{1}{2 + \frac{1}{|a_j| + 1}} \geq g_{\lambda'} \geq \frac{1}{2 + \frac{1}{|a_j| - 1}} \quad (19)$$

Now when  $j = 0$ , or  $a_j = \infty$

$$g_{\lambda'} = \frac{1}{2}. \quad (20)$$

To find approximately  $g'_{\lambda'}$  set

$$Q_i = (2, 3_k, \dots, 3_2, 3_1, a_{j+2}, 2, a, \dots, a_2, a_1). \quad (21)$$

As  $k$  increases without limit the value of  $Q_i$  may approach as nearly as we please to

$$2 - \frac{2}{3 + \sqrt{5}} = \frac{1 + \sqrt{5}}{2}. \quad (22)$$

Since, by hypothesis,  $a_n < 0$ , the value of  $g'_{\lambda'}$  may be made to approach as nearly as we please to the number  $\frac{\sqrt{5} - 1}{2}$ . Hence by a proper choice of  $i$  and  $k$  in sequence (17) the  $g_{\lambda'}$  sum  $g_{\lambda'} + g'_{\lambda'}$  may be made to approach as nearly as we please to the sum

$$\frac{1}{2} + \frac{\sqrt{5} - 1}{2} = \frac{\sqrt{5}}{2} > 1. \quad (23)$$

Hence for every value of  $\lambda'$ ,  $\frac{1}{2} < \lambda' < \tau$ , there exists an  $x_0$  such that for  $\lambda(x_0)_{\lambda'}$  there are no values of  $\lambda$  for which for all positive values of  $n_1$  and  $n$

$$Q_{n_1} = (a_{n_1}, a_{n_1-1}, \dots, a_2, a_1), \quad 0 < n_1 < n, \quad (24)$$

is a reverse  $\lambda$ -development.

When  $\tau < \lambda' < 1$  the element  $+1$  may enter a  $\lambda$ -development ( $\lambda = \lambda'$ ) when immediately preceded and followed by negative elements. Accordingly, choose the  $\lambda$ -sequence ( $\lambda = \lambda'$ )

$$(a_0, 1, a_2, a_3, a_4, a_5), \quad (25)$$

where the  $a$ 's are negative. Without difficulty we find  $g'_{\lambda'} = 1$  while  $g_{\lambda'} > 0$ . Hence

$$g_{\lambda'} + g'_{\lambda'} > 1, \quad (26)$$

and the necessary condition (6) is not satisfied. Hence for every  $\lambda'_1 \tau < \lambda' < 1$ , there exists an  $x_0$  such that for  $\lambda(x_0)_{\lambda'}$  there are no values of  $\lambda$  for which all positive values of  $n_1$  and  $n$

$$Q_{n_1} = (a_{n_1}, a_{n_1-1}, \dots, a_2, a_1) \quad 0 < n_1 < n, \quad (27)$$

is a reverse  $\lambda$ -development.

5.° When  $\lambda' = \frac{1}{2}$ , we may choose the  $\lambda$ -sequence

$$(a_0, 3_1, 3_2, \dots, 3_i, 2_1 a_{i+2}) \quad (28)$$

where  $a_0 > 0$ ,  $a_{i+2} < 0$ . To determine  $g_{\lambda'}$  set

$$Q_i = (3_i, 3_{i-1}, \dots, 3). \quad (29)$$



Let  $i = \infty$ , then

$$Q_i = \frac{3 + \sqrt{5}}{2} \quad (30)$$

and

$$g_{\lambda'} = \frac{2}{3 + \sqrt{5}} = \frac{3 - \sqrt{5}}{2}. \quad (31)$$

To determine  $g'_{\lambda'}$  set

$$Q_{i+1} = (2_1 \ 3_i, \ 3_{i-1}, \dots, \ 3_1) \quad (32)$$

and let  $i$  approach infinity. Then

$$Q_{i+1} = 2 - \frac{2}{3 + \sqrt{5}} = \frac{1 + \sqrt{5}}{2}. \quad (33)$$

Hence

$$g'_{\lambda'} = \frac{-1 + \sqrt{5}}{2}. \quad (34)$$

Therefore

$$g'_{\lambda'} + g'_{\lambda'} = 1. \quad (35)$$

The values given of  $g_{\lambda'}$  and  $g'_{\lambda'}$  are respectively the values of  $\gamma_{\lambda}$  and  $\gamma'_{\lambda'}$ . For it follows by Theorem I, §3, for  $\lambda' = \frac{1}{2}$  that  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$  when  $a_i = 2$ . Hence the sequence (29) gives the absolutely least value of  $Q_{i+1}$  for  $\text{sgn. } a_i = \text{sgn. } a_{i-1}$ ; also sequence (32) gives the least value of  $Q_{i+1}$  when  $\text{sgn. } a_{i+1} = -\text{sgn. } a_i$ . In this case the value of  $\lambda''$  is  $g'_{\lambda'} = \frac{-1 + \sqrt{5}}{2} \equiv \tau$ .

The necessary condition is also sufficient. For, when  $\lambda = \lambda'' = \tau$ , the end intervals are also normal unit intervals and consequently,  $Q_{i+1} \sim a_{i+1}$ ,  $i = 0, 1, 2, \dots, n_1$ . Further, the conditions imposed on the elements  $a$  by  $\lambda = \lambda' = \frac{1}{2}$  is precisely the same as those imposed on them by  $\lambda = \lambda'' = \tau$  with this exception that for  $\lambda' = \frac{1}{2}$   $a = \pm 2$  is always followed by a change of sign, while for  $\lambda'' = \tau$   $a = \pm 2$  is always preceded by a change of sign. Hence when the order of the elements  $a$  for  $\lambda' = \frac{1}{2}$  is reversed these elements in the reversed order satisfy the conditions imposed when  $\lambda = \lambda'' = \tau$ .

For  $\lambda = \tau_1$  choose the  $\lambda$ -sequence ( $\lambda = \lambda' = \tau$ )

$$a_0, \ 2, \ a_2, \ -2, \ a_4, \ \dots, \ a_j, \quad (36)$$

where  $a_0 < 0$ ,  $a_2 > 0$ ,  $a_4 > 0$ .

Then

$$Q_i = 2 \quad (37)$$

and

$$g_{\lambda'} = \gamma_{\lambda'} = \frac{1}{2}. \quad (38)$$

Again

$$Q_3 = (-2, \ a_2, \ 2) \quad (39)$$

and when  $a_i = \infty$ ,  $Q_3 = -2$ . Hence

$$g_{\lambda'} = \frac{1}{2}. \quad (40)$$

It may be shown that  $\gamma_{\lambda'} = g_{\lambda'}$ . Therefore by equations (38) and (40)

$$\gamma_{\lambda'} + \gamma_{\lambda'} = 1. \quad (41)$$

Here  $\lambda = \lambda'' = \gamma_{\lambda'} = \frac{1}{2}$ .

The necessary condition is in this case insufficient. For example consider the  $\lambda$ -sequence ( $\lambda = \tau$ )

$$(a_0, -2, a_2, \dots, a_n, a_{n+1}), \quad a_2 > 0 \quad (42)$$

and from the reverse development

$$Q_{n+1} = (a_{n+1}, a_n, \dots, a_2, -2). \quad (43)$$

This is not a  $\lambda$ -development ( $\lambda \equiv \lambda' = \frac{1}{2}$ ), since by theorem I, 3°, of §3,  $\lambda(x_0)_{\lambda'}$ , cannot terminate with  $\eta_1$  and in this case  $\eta_1 = -2$ , the last element in the continued fraction (43). With the exception of such reverse developments as end with the element  $\pm 2$  preceded by an element with the opposite sign, the necessary condition is also sufficient.

In all cases, therefore, except when in  $\lambda(x_0)_{\lambda'=\tau}$   $|a_1| = 2$ , while  $\text{sgn. } a_i = -\text{sgn. } a_1$ , there is for every  $\lambda(x_0)_{\lambda'=\tau}$ ,  $x_0$  real, a reverse  $\lambda$ -development ( $\lambda = \lambda'' = 1$ ) for every positive  $n_1$  and  $n$ ,  $0 < n_1 < n$ .

When  $\lambda' = 1$ ,  $g_{\lambda'} = \gamma_{\lambda'} = 1$  and  $g_{\lambda} = \gamma_{\lambda} = 0$ . The necessary condition for reverse  $\lambda$ -developments ( $\lambda = \lambda'_1 = 1$ ) is satisfied. An exception, however, must be made as in the last case. The last element in the reverse  $\lambda$ -development ( $\lambda = \lambda'' = 1$ ) must not be  $\pm 1$ .

The results obtained may be summed up in the following theorem:

**THEOREM I.** *All  $\lambda$ -developments ( $\lambda = \lambda' = \frac{1}{2}$ ) have reverse  $\lambda$ -developments ( $\lambda = \lambda'' = \tau$ )\*; all those and only those  $\lambda$ -developments ( $\lambda = \lambda' = \tau$ ) in which the absolute value  $a_1$  is not 2 when  $\text{sgn. } a_1 = -\text{sgn. } a_i$  have reverse  $\lambda$ -developments ( $\lambda = \lambda'' = \frac{1}{2}$ )\*; also, all those and only those  $\lambda$ -developments ( $\lambda = \lambda' = 1$ ) in which the absolute value  $a_1$  is not 1 have reverse  $\lambda$ -developments ( $\lambda = \lambda'' = 1$ ).*

6.° The following theorems deal with special cases of reverse  $\lambda$ -developments.

**THEOREM II.** *Every  $\lambda$ -development ( $\lambda = \lambda'$ ,  $0 < \lambda' < 1$ ) into which neither  $\pm 1$  nor  $\pm 2$  enters as an element after the first has a reverse  $\lambda$ -development ( $\lambda = \lambda' = \tau$ ).*

For every such development is a  $\lambda$ -development ( $\lambda = \lambda' = \frac{1}{2}$ ).

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\* These results are due to Hurwitz, see *Acta Math.* 12, pp. 379, 380.

THEOREM III. *Every  $\lambda$ -development ( $\lambda = \lambda'$ ,  $0 < \lambda' < 1$ ) which after the first element has only positive or only negative elements has a reverse  $\lambda$ -development.*

For, in this case,  $\gamma_{\lambda'} < 1$  and  $\gamma'_{\lambda'} = 0$ . Hence the reverse development is a  $\lambda$ -development ( $\lambda = \lambda''$ ) where

$$\lambda'' = (0) \dots 1 - \gamma_{\lambda'}.$$

THEOREM IV. *Every  $\lambda$ -development ( $\lambda = \lambda'$ ,  $0 < \lambda' < \frac{1}{2}$ ) in which, the first element aside, enter at least two consecutive twos with like signs as elements followed immediately by a change of sign has no reverse  $\lambda$ -development.*

For definiteness let at least one set of twos be positive. Then by the general theory of  $\lambda$ -developments ( $\lambda = \lambda'$ ) the element preceding the first two of the set is positive. Hence  $Q_i < 2$ , where  $i$  is the rank of the first two of the set. Hence

$$g_{\lambda'} > \frac{1}{2}. \quad (44)$$

Let  $v$  be the rank of the last two of the set. Then  $Q_v < \frac{2}{3}$ . Hence

$$g'_{\lambda'} > \frac{2}{3}; \quad (45)$$

and, consequently,

$$g_{\lambda'} + g'_{\lambda'} > 1. \quad (46)$$

Therefore, the necessary condition (6) for a reverse  $\lambda$ -development to the  $\lambda$ -developments of the description set forth in the theorem is not satisfied.

The two following theorems for very special cases are appended without demonstration.

THEOREM V. *Every  $\lambda$ -development ( $\lambda = \lambda'$ ,  $\lambda'$  the reciprocal of an integer equal to or greater than 2) in which every element just preceding a change of sign is in absolute value equal to or greater than  $\frac{1}{\lambda'} + <$  has a reverse  $\lambda$ -development.*

THEOREM VI. *Every  $\lambda$ -development ( $\lambda = \lambda'$ ,  $\lambda'$  the reciprocal of an integer equal to or greater than 2) not ending with a maximum set of twos with like signs has a reverse  $\lambda$ -development provided that whenever, the first element aside, there occurs a sequence of which each element and the following have unlike signs, the first element in this sequence is in absolute value equal to or greater than  $\frac{1}{\lambda'} + 1$ .*

## §6. A Comparison of Certain Types of Continued Fraction Developments.

1.° Since the results of this section are derived without difficulty and have no direct bearing on what follows they are given without proof.





have  $\lambda$ -developments ( $\lambda = \lambda'; II_1$ ) identical from an element of a certain rank on with  $\lambda \left( -\frac{1}{\lambda'} \right)_{\lambda'; II_1}$ , or with  $\lambda \left( \frac{1}{\lambda'} \right)_{\lambda'; II_1}$ .

3.° Continued fractions based on defining equation (2).

THEOREM III. Only for those numbers whose  $\lambda$ -developments ( $\lambda = \lambda'; I_2$ ) from an element of a certain rank on are identical with  $\lambda \left( \frac{1}{1-\lambda'} \right)_{\lambda'; I_2}$  or  $\lambda \left( -\frac{1}{1-\lambda'} \right)_{\lambda'; I_2}$ , do the  $\lambda$ -developments ( $\lambda = \lambda'; I_2$ ) differ from the corresponding  $\lambda$ -developments ( $\lambda = \lambda'; II_2$ ).

THEOREM IV. The number whose  $\lambda$ -developments ( $\lambda = \lambda'; I_2$ ) from an element of a certain rank on are identical with  $\lambda \left( \frac{1}{1-\lambda'} \right)_{\lambda'; I_2}$ , or with  $\lambda \left( -\frac{1}{1-\lambda'} \right)_{\lambda'; I_2}$  have  $\lambda$ -developments ( $\lambda = \lambda', II_2$ ) identical from an element of a certain rank on with  $\lambda \left( -\frac{1}{\lambda'} \right)_{\lambda'; II_2}$ , or with  $\lambda \left( \frac{1}{\lambda'} \right)_{\lambda'; II_2}$ .

4.° Comparison of continued fractions arising from defining equations (1) and (2).

THEOREM V. When for either interval system and for  $\lambda = \lambda'$   $x_0$  is expressed as a continued fraction based on defining equation (1), this continued fraction differs from that of  $x_0$  in the same interval system and for the same  $\lambda$ ,  $\lambda = \lambda'$ , but based on defining equation (2) only in this: the elements of odd rank in the first continued fraction appear with opposite signs in the second continued fraction.

REMARK I. The continued fraction development of a positive number for  $\lambda' = 1$  and based on defining equation (2) and interval system  $I_1$  is identically the simple continued fraction representing the number.

REMARK II. The periodicity of continued fractions ( $\lambda = \lambda'$ ) representing quadratic irrationalities when these continued fractions belong to one of the types considered in this section implies in the remaining types the periodicity of the continued fractions representing quadratic irrationalities.

## §7. Equivalence.

1.° In simple continued fractions the equivalence, proper or improper, of two real numbers appears in the ultimate likeness\* of the continued fractions representing the numbers. In  $\lambda$ -developments ( $\lambda = \lambda'$ ) a distinction is made

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\* To avoid a special case it is here assumed that the last element in the continued fraction representing a rational number  $\infty$ .

between proper and improper equivalence except when the two numbers in question are both properly and improperly equivalent. The relation of the  $\lambda$ -developments ( $\lambda = \lambda'$ ) of two properly or improperly equivalent numbers is further dependent on the particular value  $\lambda'$  assigned to  $\lambda$ .

The  $\lambda$ -developments ( $\lambda = \lambda', \tau < \lambda' \leq 1$ ) of two properly equivalent numbers are ultimately alike; those of two improperly equivalent numbers differ ultimately in the signs of the elements and in this respect only.

For  $\lambda$ -developments ( $\lambda = \lambda'; 0 < \lambda' \leq \tau$ ) two cases arise. Let  $x'_0$  be properly equivalent to  $x_0$ . Then a)  $\lambda(x'_0)_{\lambda'}$  is ultimately like  $\lambda(x_0)_{\lambda'}$ ; or b)  $\lambda(x'_0)_{\lambda'}$  is ultimately like a  $\lambda$ -development ( $\lambda = \lambda'$ ) derived by a simple rule (to be explained hereafter) from  $\lambda(x_0)_{\lambda'}$  and called the *associate* of  $\lambda(x_0)_{\lambda'}$ . To cover the case of the improper equivalence of  $x'_0$  and  $x_0$  we need in the preceding statement merely replace  $\lambda(x_0)_{\lambda'}$  by  $\lambda(-x_0)_{\lambda'}$ .

A very special case of b) appears in  $H_I$  and  $H_{II}^*$  but hardly in such a way as to make its real significance obvious. In the present general theory b) assumes an important role and shows in a particularly interesting way how the relation of  $\lambda(x'_0)_{\lambda = \lambda'}$  to  $\lambda(x_0)_{\lambda = \lambda}$  depends on the relation of the values of the ultimate elements in  $\lambda(x_0)_{\lambda = \lambda'}$  to the particular value of  $\lambda'$  assigned to  $\lambda$ .

2.° Two real numbers  $x_0$  and  $x'_0$  are said to be properly equivalent when they satisfy an equation of the form

$$x'_0 = \frac{\alpha x_0 - \beta}{\gamma x_0 - \delta} \quad (1)$$

in which  $\alpha, \beta, \gamma, \delta$ , are integers such that

$$\beta\gamma - \alpha\delta = 1. \quad (2)$$

THEOREM I.† *When in the  $\lambda$ -developments of two real numbers  $x_0$  and  $x'_0$  for the same or for different values of  $\lambda$ ; viz., in*

$$\begin{aligned} \lambda(x_0) &= (a_0, a_1, \dots, a_{n-1}, x_n), \\ \lambda(x'_0) &= (a'_0, a'_1, \dots, a'_{m-1}, x'_m), \end{aligned}$$

*$x_m = x_n$  for some values of  $m$  and  $n$ , then  $x_0$  and  $x'_0$  are properly equivalent.*

For, in the usual notation

$$x_0 = \left( \frac{p}{r}, -\frac{q}{s} \right) x_n, \quad (3)$$

$$x'_0 = \left( \frac{p'}{r'}, -\frac{q'}{s'} \right) x'_m, \quad (4)$$

\* Hurwitz: *loc. cit.*

† Cf. Hurwitz: *loc. cit.*

where  $p, q, r, s, p', q', r', s'$  are integers satisfying the relation

$$\left| \begin{matrix} p, & -q \\ r, & -s \end{matrix} \right| = \left| \begin{matrix} p', & -q' \\ r', & -s' \end{matrix} \right| = 1.$$

From equations (3) and (4), since  $x'_m = x_n$

$$x'_0 = \left( \begin{matrix} \alpha, & -\beta \\ \gamma, & -\delta \end{matrix} \right) x_0, \quad (5)$$

where  $\alpha, \beta, \gamma, \delta$  are integers depending on  $p, q, r, s, p', q', r', s'$  and satisfying equation (2).

3.° Given that  $x_0$  and  $x'_0$ , two real numbers, are properly equivalent, we seek to determine the relation of their  $\lambda$ -developments ( $\lambda = \lambda'$ ). In what follows  $x_0$  and  $x'_0$  are by hypothesis irrational; the results become applicable to rational numbers on adopting the convention that every element or partial denominator in their  $\lambda$ -developments ( $\lambda = \lambda'$ ) from those of a certain rank on is infinite.

When  $x_0$  and  $x'_0$  are related as in equation (1) we shall say that  $x_0$  passes into  $x'_0$  by the proper unitary substitution

$$\left( \begin{matrix} \alpha_1 & -\beta \\ \gamma_1 & -\gamma \end{matrix} \right).$$

Let  $S$  and  $T$  be defined by the following equations:

$$S_{(x)} \equiv x + 1, \quad T_{(x)} \equiv -\frac{1}{x}. \quad (6)$$

Then  $S$  and  $T$  denote proper unitary substitution and are called elementary substitutions.

The theory of equivalence here developed is based on the well-known proposition.\*

**THEOREM II.** *Every proper unitary substitution is a product of the elementary substitutions  $S$  and  $T$ .*

4.° We shall determine directly the transformations which  $\lambda(x_0)_{\lambda'}$  undergoes when  $S$  and  $T$  separately operate on  $x_0$ , and from these results derive by means of theorem II the changes produced in  $\lambda(x_0)_{\lambda'}$  when  $x_0$  is operated on by any proper unitary substitution.

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\* Cf. Mathews: Theory of Numbers, Part I, Art. 112; or Jordan: Cours d'analyse, 2d. Edn., Vol. II, pp. 327-330; also Klein: Theorie der elliptischen Modulfunktionen, I, pp. 218, 219.

The theory of equivalence was worked out following the method of Hurwitz. The present method as probably offering a simpler solution was taken up at the suggestion of Prof. E. H. Moore.

THEOREM III. For  $x_0$  real  $\lambda(x_0)_{\lambda'}$  and  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  from elements of a certain rank on are identical except when  $\lambda' < |x_0| < \frac{1}{\lambda'}$ ,  $0 < \lambda' \leq \tau$ , and the complete quotients beginning with  $x_1$  all have the same sign and are in absolute value equal to or less than  $\frac{1}{\lambda'} + 1$ .

Let it be assumed that

$$\lambda(x_0)_{\lambda'} = (a_0, a_1, \dots, a_{n-1}, x_n). \quad (7)$$

Then

$$x_0 = a_0 - \frac{1}{x_1} = \frac{a_0 x_1 - 1}{x_1} \quad (8)$$

and

$$-\frac{1}{x_0} = \frac{x_1}{1 - a_0 x_1}. \quad (9)$$

There are several cases to consider.

$$\text{a) } |x_0| > \frac{1}{\lambda'}.$$

In this case

$$-\frac{1}{x_0} \sim 0 \quad (10)$$

and

$$-\frac{1}{x_0} = 0 - \frac{1}{a_0 - \frac{1}{x_1}} = (0, a_0, x_1). \quad (11)$$

Therefore

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (0, a_0, a_1, \dots, a_{n-1}, x_n). \quad (12)$$

$$\text{b) } |x_0| < \lambda'.$$

In this case

$$x_0 \sim 0 \quad (13)$$

and

$$-\frac{1}{x_0} = x_1. \quad (14)$$

Therefore

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (a_1, a_2, \dots, a_{n-1}, x_n) \quad (15)$$

$$\text{c) } |x_0| = \lambda', |x_0| = \frac{1}{\lambda'}.$$



These cases have been dealt with fully in § 2. The results were entirely in accord with theorem III.

$$d) \quad \lambda' < |x_0| < \frac{1}{\lambda'}.$$

It is sufficient to consider the case in which  $x_0 > 0$ . The result for  $x_0 < 0$  can be obtained at once from the symmetry of the interval system

$$d_1) \quad a_0 = 1.$$

Then

$$x_0 = 1 - \frac{1}{x_1} \quad (16)$$

and

$$-\frac{1}{x_0} = -1 - \frac{1}{x_1 - 1}. \quad (17)$$

When  $-\frac{1}{x_0} \sim -1$ , as is always true for  $\tau < \lambda' \leq 1$ , then

$$\lambda \left( -\frac{1}{x_0} \right)_{\lambda'} = (-1, x_1 - 1) = (-1, a_1 - 1, a_2, \dots, a_{n-1}, x_n). \quad (18)$$

and

$$x_1 = \left( \frac{1}{\lambda'} + 1 \right) \dots - \frac{\lambda'}{1 - \lambda'}.$$

Since the normal interval for  $x_1$  is

$$x_1 = \frac{1}{1 - \lambda'} \dots \left( -\frac{1}{\lambda'} \right),$$

there remains for consideration the two cases:

$$x_1 = \left( -\frac{\lambda'}{1 - \lambda'} \right) \dots \left( -\frac{1}{\lambda'} \right), \quad x_1 < 0. \quad (19)$$

and

$$x_1 = \frac{1}{1 - \lambda'} \dots \frac{1}{\lambda'} + 1. \quad (20)$$

The first of this, (19), implies that  $\tau < \lambda' \leq 1$  and is consequently included in the case  $-\frac{1}{x_0} \sim -1$ . For the second case, (20),

$$\frac{1}{1 - \lambda'} < x_1 < \frac{1}{\lambda'} + 1$$

it follows that  $0 < \lambda' \leq \tau$ . The complete treatment of this case is deferred until a preliminary examination has been made of the other cases.

$$d_2) \quad x_0 > 0, \quad a_0 > 1.$$

Without at all limiting the generality of this case it may be assumed that

$$a_1 = a_2 = \dots = a_h = 2 \neq a_{h+1}, \quad h \text{ a positive integer.}$$

In this case\*

$$\frac{1}{x_0} \sim 1. \quad (21)$$

Now

$$-\frac{1}{x_0} \equiv -1 + \frac{x_0 - 1}{x_0} \equiv -1 + \frac{1}{1 + \frac{1}{x_0 - 1}}. \quad (22)$$

Hence

$$\lambda \left( -\frac{1}{x_0} \right)_{\lambda'} = \left( -1, -1 - \frac{1}{x_0 - 1} \right). \quad (23)$$

Since  $x_0 < \frac{1}{\lambda'}$  then

$$1 + \frac{1}{x_0 - 1} > 1 + \lambda'$$

and in consequence  $1 + \frac{1}{x_0 - 1}$  corresponds to 2 or to some integer greater than 2. In this case we may say that

$$\lambda \left( 1 + \frac{1}{x_0 - 1} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{a_0 - 2}, 2 + \frac{1}{x_0 - 1} \right). \quad (24)$$

To prove this it is necessary to show that

$$2_i - \frac{x_0 - i - 1}{x_0 - i} \sim 2_i, \quad i = 1, 2, \dots, a_0 - 2. \quad (25)$$

Now

$$2_i - \frac{x_0 - i - 1}{x_0 - i} = 1 + \frac{1}{x_0 - i}. \quad (26)$$

It is to be shown, therefore, that

$$1 + \frac{1}{x_0 - i} = 1 + \lambda' \dots (2 + \lambda'), \quad i = 1, 2, \dots, a_0 - 2. \quad (27)$$

Since

$$x_0 < \frac{1}{\lambda'}$$

then, certainly,

$$0 < x_0 - i < \frac{1}{\lambda'}, \quad 1 \leq i \leq a_0 - 2,$$

and, consequently,

$$1 + \frac{1}{x_0 - i} > 1 + \lambda'. \quad (28)$$

\* When  $\tau < \lambda' < 1$ ,  $h = 0$ . But all cases in which  $\tau < \lambda' < 1$  are disposed of completely under  $d_1$ ). We may say that  $d_2$ ) includes  $d_1$ ) under the hypothesis that  $h = 0$ , and removing the restriction on  $a_0$ .

On the other hand

$$x_0 - i \geq x_0 - a_0 + 2 = 1 + \lambda' \dots (2 + \lambda'). \quad (29)$$

Hence

$$\frac{1}{x_0 - i} < 1.$$

and

$$1 + \frac{1}{x_0 - i} < 2 < 2 + \lambda'. \quad (30)$$

By the inequalities (28) and (30) follow the interval equation (27) and thence relation (25).

Hence, by successive reduction, we get

$$\lambda \left( 1 + \frac{1}{x_0 - 1} \right)_{\lambda'} = \left( 2_1, 2_2, \dots, 2_{a_0-2}, \frac{x_0 - a_0 + 2}{x_0 - a_0 + 1} \right). \quad (31)$$

By virtue of the relation

$$x_0 - a_0 = -\frac{1}{x_1}$$

equation (31) reduces to equation (24). Combining equations (23) and (24)

$$\lambda \left( -\frac{1}{x_0} \right)_{\lambda'} = \left( -1, -2_1, -2_2, \dots, -2_{a_0-2}, -2 - \frac{1}{x_1 - 1} \right). \quad (32)$$

When

$$-2 - \frac{1}{x_1 - 1} \sim -2,$$

then

$$\begin{aligned} \lambda \left( -\frac{1}{x_0} \right)_{\lambda'} &= (-1, -2_1, -2_2, \dots, -2_{a_0-2}, -2_{a_0-1}, x_1 - 1) \\ &= (-1, -2_1, -2_2, \dots, -2_{a_0-2}, -2_{a_0-1}, a_1 - 1, a_2, \dots, a_{n-1}, x_n). \end{aligned} \quad (33)$$

When  $-2 - \frac{1}{x_1 - 1}$  does not correspond to  $-2$ , a further reduction is necessary.

$$d_3) \quad x_0 > 0, a_0 \geq 1, 0 < \lambda' \leq \tau.$$

To include in one investigation the deferred case under  $d_1$ ) and that under  $d_2$ ) we may study  $\lambda \left( a + \frac{1}{x_1 - 1} \right)_{\lambda'}$ ,  $a > 0$ , when  $a + \frac{1}{x_1 - 1}$  does not correspond to  $a$ . By means of the symmetry of the interval system we then derive at once  $\lambda \left( -a - \frac{1}{x_1 - 1} \right)_{\lambda'}$ . When  $a = 1$ , this gives the deferred case under  $d_1$ ) and when  $a = 2$ , that under  $d_2$ ).

Observe first that  $a + \frac{1}{x_1 - 1}$  does not correspond to  $a - 1$ . For when  $a = 1$  this would make zero an element following the first in  $\lambda\left(-\frac{1}{x_0}\right)$ , which is not possible. When  $a = 2$  we should have 1 as an element not the first in  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  and hence  $\tau < \lambda' < 1$ . But the case for  $\tau < \lambda' < 1$  was completely disposed of under  $d_1$ ). Hence we study  $\lambda\left(a + \frac{1}{x_1 - 1}\right)_{\lambda'}$ ,  $0 < \lambda' < \tau$ , for those cases in which  $a + \frac{1}{x_1 - 1} \sim a + 1$  or to some greater integer.

$$\text{In this case} \quad a + \frac{1}{x_1 - 1} > a + \lambda', \quad 0 < \lambda' < \tau, \quad (34)$$

$$\text{and in consequence} \quad 0 < x_1 < \frac{1}{\lambda'} + 1. \quad (35)$$

Hence  $\text{sgn. } x_1 = \text{sgn. } a_0 = +1$ . Therefore the condition (35) on  $x_1$  may be replaced by

$$\frac{1}{1 - \lambda'} < x_1 < \frac{1}{\lambda'} + 1. \quad (36)$$

It is now assumed for both deferred cases under  $d_1$ ) and  $d_2$ ) that

$$a_1 = a_2 = \dots = a_h = 2 \neq a_{h+1}, \quad (37)$$

$h$  a positive integer. On this hypothesis

$$x_i = 2 - \frac{1}{x_{i+1}} \quad i = 1, 2, \dots, h \quad (38)$$

and, consequently,

$$\frac{1}{1 - \lambda'} < x_i < \frac{1}{\lambda'} + 1, \quad i = 1, 2, \dots, h. \quad (39)$$

By equation (38) it is evident that

$$a + \frac{1}{x_1 - 1} = a + 1 + \frac{1}{x_2 - 1} = a + 2 + \frac{1}{x_3 - 1} = \dots = a + h + \frac{1}{x_{h+1} - 1}. \quad (40)$$

The further reduction of this expression will be made in two subcases.

$$d_{3_1}) \quad a + h + \frac{1}{x_{h+1} - 1} \sim a + h.$$

Then

$$\lambda\left(a + \frac{1}{x_1 - 1}\right)_{\lambda' (0 < \lambda' < \tau)} = (a + h_1 - x_{h+1} + 1) = (a + h, -a_{h+1} + 1, -a_{h+2}, \dots, -a_{h-1}, -x_n). \quad (41)$$

$$d_{3_2}) \quad a + h + \frac{1}{x_{h+1} - 1} \text{ does not correspond to } a + h.$$



In this case  $a + h + \frac{1}{x_{h+1}-1}$  does not correspond to  $a + h - 1$  or to a lesser integer. For, did the correspondence hold, then

$$a + h + \frac{1}{x_{h+1}-1} < a + h - 1 + \lambda', \quad (42)$$

whence it follows that  $x_{h+1} < 0$  and further

$$|x_{h+1}| > \frac{1}{\lambda'}. \quad (43)$$

Hence

$$a + h + \frac{1}{x_{h+1}-1} > a + h - \frac{\lambda'}{1+\lambda'}. \quad (44)$$

Now  $a + h - \frac{\lambda'}{1+\lambda'} < a + h - 1 + \lambda'$  only when  $\lambda' > \tau$ , values of  $\lambda'$  which do not here come under consideration.

From the cases considered we conclude that  $x_{h+1} > 0$  and  $a + h + \frac{1}{x_{h+1}-1} \sim a + h + 1$ , or to a greater integer. Since  $x_{h+1} = 2 + \lambda'$ ,  $a + h + \frac{1}{x_{h+1}-1} < a + h + 1$ . Therefore

$$a + h + \frac{1}{x_{h+1}-1} \sim a + h + 1. \quad (45)$$

And since

$$a + h + \frac{1}{x_{h+1}-1} = a + h + 1 - \frac{x_{h+1}-2}{x_{h+1}-1}, \quad (46)$$

$$\lambda \left( a + \frac{1}{x_1-1} \right)_{\lambda'} = \left( a + h + 1, \frac{x_{h+1}-1}{x_{h+1}-2} \right), \frac{1}{1-\lambda'} < x_{h+1} < \frac{1}{\lambda'} + 1. \quad (47)$$

The continuation of this  $\lambda$ -development ( $\lambda = \lambda'$ ) comes directly under formula (24) so that it follows at once that

$$\lambda \left( a + \frac{1}{x_1-1} \right)_{\lambda'} = \left( a + h + 1, 2_1, 2_2, \dots, 2_{a-h-1}, 2 + \frac{1}{x_{h+2}-1} \right). \quad (48)$$

It remains only to repeat on  $2 + \frac{1}{x_{h+2}-1}$  the process applied to  $a + \frac{1}{x_1-1}$  to get a further development of  $a + \frac{1}{x_1-1}$ . The repeated application of this process evidently leads to a  $\lambda \left( a + \frac{1}{x_1-1} \right)_{\lambda'}$  of one of two types. Either the last

element in  $\lambda\left(a + \frac{1}{x_1-1}\right)_{\lambda'}$  corresponds to  $+2$ , in which case the  $\lambda$ -development (48) ultimately coincides with the  $\lambda$ -development (44), or it is true that

$$\frac{1}{1-\lambda'} \equiv x_i \equiv \frac{1}{\lambda'} + 1, \quad i = 1, 2, \dots, n,$$

$n$  any positive integer however great, and this latter case the  $\lambda$ -development is of the type (48).

From this result  $\lambda\left(-a - \frac{1}{x_1-1}\right)_{\lambda'}$  is at once completely determined so that the  $\lambda$ -developments in the deferred cases under  $d_1$ ) and  $d_2$ ) may be at once written down.

The conclusions reached under a), b), c), d), bearing in mind the symmetry of the interval system, completely establishes theorem III.

5.° When  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  are not ultimately alike, then by combining equations (17) and (18) on the one hand and equations (23) and (48) on the other we get the following two forms of  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$ ,  $x_0 > 0$ : viz.

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = \left(-a-h-1, -2_1, -2_2, \dots, -2_{a-h-1}, -2 - \frac{1}{x_{h+2}-1}\right),$$

$$a = 1, x_{h+2} = \frac{1}{1-\lambda'} \dots \frac{1}{\lambda'} + 1; \quad (49)$$

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = \left(-1, -2_1, -2_2, \dots, -2_{a_0-2}, -a-h-1, -2_1, -2_2, \dots, -2_{a-h-1}, -2 - \frac{1}{x_{h+2}-1}\right),$$

$$a = 2, x_{h+2} = \frac{1}{1-\lambda'} \dots \frac{1}{\lambda'} + 1. \quad (50)$$

The two forms of  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  represented by equations (49) and (50) may be reduced to one. Let the convention be adopted that when  $a_0 = 1$  and, therefore,  $a = 1$ , then in equation (50) the elements  $-1, -2_1, -2_2, \dots, -2_{a_0-2}$  which precede the element  $-a-h-1$  simply do not appear. Under this convention equations (49) and (50) may both be represented by the single equation

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = \left(-1, -2_1, -2_2, \dots, -2_{a_0-2}, -a-h-1, -2_1, \dots, -2_{a-h-1}, -2 - \frac{1}{x_{h+2}-1}\right), \quad (51)$$

where

$$x_{a+2} = \frac{1}{1-\lambda'} \cdots \frac{1}{\lambda'} + 1,$$

and  $a = 1$  or  $2$  according  $a_0 = 1$  or  $a_0 > 1$ .

When  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$ ,  $\lambda_0 > 0$ ,  $0 < \lambda \leq \tau$ , are not ultimately alike\* then by the law already applied in equation (51) we have

$$\lambda\left(-\frac{1}{x_0}\right)_{\lambda'} = (-1, -2, \dots, -2_{a_0-2}, -a-h-1, -2_1, \dots, \\ -2_{\frac{a-3}{h+1}}, -3-h_1, -2_1, \dots, -2_{\frac{a-3}{h+h_1+2}}, -3-h_2, \dots), \quad (52)$$

where  $h$  is defined by the condition

$$\alpha_1 = \alpha_2 = \dots = \alpha_h = 2 \neq \alpha_{h+1},$$

$h_1$  by the condition

$$\alpha_{h+2} = \alpha_{h+3} = \dots = \alpha_{h+h_1+1} = 2 \neq \alpha_{h+h_1+2},$$

with similar definitions for  $h_2, h_3, \dots$ , and where for  $a_0 \geq 2$ ,  $a = 2$ ; and for  $a_0 = 1$ ,  $a = 1$  and the elements  $-1, -2_1, \dots, -2_{a_0-2}$  immediately preceding the element  $-a-h-1$  do not appear in  $\lambda(x_0)_{\lambda'}$ .

When  $x_0 < 0$ , then  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  is found by changing the signs of all the elements in equation (52) with the analogous definitions of  $a$  and  $h$ .

When  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  are not ultimately alike,  $\lambda\left(-\frac{1}{x_0}\right)_{\lambda'}$  is called the *associate* of  $\lambda(x_0)_{\lambda'}$ .

Since the value and order of the ultimate elements in the associate depends only on the value and order of the ultimate elements in the  $\lambda$ -development ( $\lambda = \lambda'$ ), an obvious inference from equation (52), we have the following proposition.

**COROLLARY.** *If  $\lambda(y_0)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  are ultimately alike and  $\lambda(y_0)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  have associated developments, then the associate of  $\lambda(y_0)_{\lambda'}$  is ultimately like the associate of  $\lambda(x_0)_{\lambda'}$ .*

In view of the results now obtained Theorem III may take the form:

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\* The phrase "ultimately alike" is used here and elsewhere in the sense of the longer phrase "identical from elements of certain ranks on."

THEOREM IV. For  $x_0$  real  $\lambda \left( -\frac{1}{x_0} \right)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  are ultimately alike or  $\lambda \left( -\frac{1}{x_0} \right)_{\lambda'}$  and the associate of  $\lambda(x_0)_{\lambda'}$  are ultimately alike, and in the latter case  $0 < \lambda' < \tau$ ,  $\lambda' < |x_0| < \frac{1}{\lambda'} + 1$ ,  $|x_i| < \frac{1}{\lambda'} + 1$ , and  $\text{sgn } x_{i+1} = \text{sgn } x_i$   $i = 1, 2, \dots, n-1$ .

6.° In the case of the substitution  $S$  the following theorem is true:

THEOREM V. For  $x_0$  real  $\lambda(x_0 + 1)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  are ultimately alike, except when for  $0 < \lambda' < \tau$ , either  $x_0 + 1 \sim 0$  and  $x_0 \sim 0$ , or  $x_0 + 1 \sim 1$  and  $x_0 \sim -1$ , and beginning with  $x_1$  all complete quotients in  $\lambda(x_0)_{\lambda'}$  have the same sign and are in absolute value equal to or less than  $\frac{1}{\lambda'} + 1$  and in this latter case  $\lambda(x_0 + 1)_{\lambda'}$  and the associate of  $\lambda(x_0)_{\lambda'}$  are ultimately alike.

As in the demonstration of theorem III it is convenient to consider separately the several cases which may arise.

$$\text{a) } x_0 = 0 \dots -1.$$

In this case it is obvious that

$$x_0 + 1 = a_0 + 1 - \frac{1}{x_1}$$

and that

$$x_0 + 1 \sim a_0 + 1.$$

Hence

$$\lambda(x_0 + 1)_{\lambda'} = (a_0 + 1, a_1, \dots, a_{n-1}, x_n). \quad (53)$$

$$\text{b) } x_0 = (-1) \dots -\lambda'; \quad x_0 + 1 = (0) \dots (\lambda').$$

By hypothesis

$$x_0 = -1 - \frac{1}{x_1}$$

and, consequently,

$$x_0 + 1 = -\frac{1}{x_1} \sim 0.$$

Therefore

$$\lambda(x_0 + 1)_{\lambda'} = (0, a_1, a_2, \dots, a_{n-1}, x_n). \quad (54)$$

$$\text{c) } x_0 = (-1) \dots -\lambda', \quad x_0 + 1 = \lambda' \dots 1.$$

Again

$$x_0 = -1 - \frac{1}{x_1}$$

and

$$x_0 + 1 = -\frac{1}{x_1} = 1 - \frac{x_1 + 1}{x_1}.$$



Since  $x_0 + 1 \sim 1$ , then

$$\lambda(x_0 + 1)_{\lambda'} = \left(1, 1 - \frac{1}{x_1 - 1}\right). \quad (55)$$

But inasmuch as  $x_0 + 1 \sim 1$ , then  $x_1 < 0$ ; also, the last equation may be written

$$\lambda(x_0 + 1)_{\lambda'} = \left(1, 1 + \frac{1}{-x_1 - 1}\right),$$

where now  $-x_1$  corresponds to  $+x_1$  in equation (17). Hence applying the conclusion reached in equation (17), we may say that  $\lambda(x_0 + 1)_{\lambda'}$  is ultimately like  $\lambda(x_0)_{\lambda'}$  or ultimately like the associate of  $\lambda(x_0)_{\lambda'}$ , a conclusion agreeing with theorem V.

$$d) \quad x_0 = (-\lambda') \dots (0), \quad x_0 + 1 = 0 \dots (\lambda').$$

Here

$$x_0 = -\frac{1}{x_1}$$

and

$$x_0 + 1 = \frac{x_1 - 1}{x_1} = 0 + \frac{1}{1 + \frac{1}{x_1 - 1}}.$$

Since

$$x_0 + 1 \sim 0$$

then

$$\lambda(x_0 + 1)_{\lambda'} = \left(0, -1 - \frac{1}{x_1 - 1}\right). \quad (56)$$

The continuation of this  $\lambda$ -development ( $\lambda = \lambda'$ ) presents precisely the problem dealt with under equation (17) and hence the conclusions of theorem III apply to equation (56), hence, also, theorem V holds for equation (56).

$$e) \quad x_0 = (-\lambda') \dots 0, \quad x_0 + 1 = \lambda' \dots 1.$$

In this case

$$x_0 = -\frac{1}{x_1}$$

and

$$x_0 + 1 = 1 - \frac{1}{x_1}.$$

Hence since

$$x_0 + 1 \sim 1$$

then

$$\lambda(x_0 + 1)_{\lambda'} = (1, a_1, a_2, \dots, a_{n-1}, x_n). \quad (57)$$

The conclusions reached under a), b), c), d), e), establish theorem V.

7.° Denote the associate of  $\lambda(x_0)_{\lambda'}$  by  $A\lambda(x_0)_{\lambda'}$  and the associate of  $A\lambda(x_0)_{\lambda'}$  by  $AA\lambda(x_0)_{\lambda'}$ . For brevity let the symbol  $\asymp$  stand for the phrase "is ultimately like." Further read the symbolic expression

$$\lambda(y_0)_{\lambda'} \asymp \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}$$

disjunctively, thus:—  $\lambda(y_0)_{\lambda'}$  is ultimately like  $\lambda(x_0)_{\lambda'}$  or ultimately like  $A\lambda(x_0)_{\lambda'}$ .

THEOREM VI.  $AA\lambda(x_0)_{\lambda'}$  is ultimately like  $\lambda(x_0)_{\lambda'}$ .

Under certain conditions given in theorem VI.

$$\lambda(T(x_0))_{\lambda'} \asymp A\lambda(x_0)_{\lambda'}.$$

Hence under these conditions since the value and order of the ultimate elements in the associate depends only on the value and order of the ultimate elements in the original  $\lambda$ -development (corollary under theorem III), we have

$$\lambda(TT(x_0))_{\lambda'} \asymp \frac{A\lambda(x_0)_{\lambda'}}{AA\lambda(x_0)_{\lambda'}}.$$

But since  $T^2$  is the identical substitution

$$\lambda(TT(x_0))_{\lambda'} \asymp \lambda(x_0)_{\lambda'}.$$

Comparing with the preceding equation

$$AA\lambda(x_0)_{\lambda'} = \lambda(x_0)_{\lambda'}.$$

Let  $P$  denote a product of substitutions made up of the factors  $S$  and  $T$ . Then

THEOREM VII. If  $\lambda(P(x_0))_{\lambda'} \asymp \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}$ , then  $\lambda(TP(x_0))_{\lambda'} \asymp \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}$ .

For, by theorem IV.

$$\lambda(TP(x_0))_{\lambda'} \asymp \frac{\lambda(P(x_0))_{\lambda'}}{A\lambda(P(x_0))_{\lambda'}}.$$

$$\text{a) } \lambda(TP(x_0))_{\lambda'} \asymp \lambda(P(x_0))_{\lambda'}.$$

In this case we have at once by virtue of our hypothesis

$$\lambda(TP(x_0))_{\lambda'} \asymp \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}.$$

$$\text{b) } \lambda(TP(x_0))_{\lambda'} \asymp A\lambda(P(x_0))_{\lambda'}.$$

Since the value and order of the ultimate elements in the associate depends only on the value and order of the ultimate elements in the original  $\lambda$ -development (corollary under theorem III),

$$A\lambda(P(x_0))_{\lambda'} \asymp \frac{A\lambda(x_0)_{\lambda'}}{AA\lambda(x_0)_{\lambda'}} \asymp \frac{A\lambda(x_0)_{\lambda'}}{\lambda(x_0)_{\lambda'}} \text{ by theorem VI.}$$

Hence

$$\lambda (TP(x))_{\lambda'} = \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}.$$

THEOREM VIII. If  $\lambda(P(x_0))_{\lambda'} = \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}$ , then  $\lambda(SP(x_0))_{\lambda'} = \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}$ .

For, by theorem V.

$$\lambda(SP(x_0))_{\lambda'} = \frac{\lambda(P(x_0))_{\lambda'}}{A\lambda(P(x_0))_{\lambda'}}.$$

$$a) \quad \lambda(SP(x_0))_{\lambda'} = \lambda(P(x_0))_{\lambda'}.$$

By an immediate inference from the hypothesis

$$\lambda(SP(x_0))_{\lambda'} = \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}.$$

$$b) \quad \lambda(SP(x_0))_{\lambda'} = A\lambda(P(x_0))_{\lambda'}.$$

By the corollary under theorem III we have

$$A\lambda(P(x_0))_{\lambda'} = \frac{A\lambda(x_0)_{\lambda'}}{AA\lambda(x_0)_{\lambda'}} = \frac{A\lambda(x_0)_{\lambda'}}{\lambda(x_0)_{\lambda'}}, \text{ by theorem VI.}$$

Therefore

$$\lambda(SP(x_0))_{\lambda'} = \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}.$$

Theorems IV, V, VII and VIII provide a basis for the application of the method of complete induction. They justify the following theorem:

$$\text{THEOREM IX.} \quad \lambda(P(x_0))_{\lambda'} = \frac{\lambda(x_0)_{\lambda'}}{A\lambda(x_0)_{\lambda'}}.$$

By theorem II every proper unitary substitution is a product of the powers of the elementary substitutions  $S$  and  $T$ . Hence our results may be summed up as follows:

THEOREM X. When for  $x_0$  real  $\alpha, \beta, \gamma, \delta$  are integers satisfying the relation

$$\beta\gamma - \alpha\delta = 1$$

and

$$x'_0 = \frac{\alpha x_0 - \beta}{\gamma x_0 - \delta},$$

then  $\lambda(x'_0)_{\lambda'}$  and  $\lambda(x_0)_{\lambda'}$  are ultimately alike or  $\lambda(x'_0)_{\lambda'}$  and  $A\lambda(x_0)_{\lambda'}$  are ultimately alike, and in the latter case  $0 < \lambda' < \tau$  and from a certain rank on the complete quotients in  $\lambda(x_0)_{\lambda'}$  all have the same sign and are in absolute value equal to or less

than  $\frac{1}{\lambda'} + 1$ .

8.° Since  $x_0$  and  $-x_0$  are improperly equivalent and  $\lambda(x_0)_{\lambda'}$  and  $\lambda(-x_0)_{\lambda'}$  differ in and only in the signs of their respective elements, the case of improper equivalence may be disposed of at once as in the following theorem:

THEOREM XI. *When  $x'_0$  and  $x_0$ , two real irrational numbers, are improperly equivalent, then  $\lambda(x'_0)_{\lambda'}$  and  $\lambda(-x_0)$  are ultimately alike, or  $\lambda(x'_0)_{\lambda'}$  and  $A\lambda(-x_0)_{\lambda'}$  are ultimately alike, and in the latter case  $0 < \lambda' < \tau$  and from a certain rank on the complete quotients in  $\lambda(x_0)_{\lambda'}$  all have the same sign and are in absolute value equal to or less than  $\frac{1}{\lambda'} + 1$ .*

9.° We add here some numerical examples of  $\lambda$ -developments ( $\lambda = \lambda'$ ) having associate  $\lambda$ -developments  $\lambda = \lambda'$ . The value of  $x_0$  is first given followed by its  $\lambda$ -development ( $\lambda = \lambda'$ ), then a corresponding value  $x'_0$  whose  $\lambda$ -development ( $\lambda = \lambda'$ ) is ultimately like the associate of  $\lambda(x_0)_{\lambda'}$  with its  $\lambda$ -development ( $\lambda = \lambda'$ ).

$$\lambda' = \frac{1}{2}.$$

$$x_0 = \frac{3 + \sqrt{5}}{2}, \lambda(x_0)_{\lambda'} = (3; 3; 3; \dots),$$

$$x'_0 = -\frac{3 + \sqrt{5}}{2}, \lambda(x'_0)_{\lambda'} = (-3; -3; -3; \dots).$$

$$\lambda' = \frac{1}{3}.$$

$$x_0 = \frac{2 + \sqrt{2}}{2}, \lambda(x_0)_{\lambda'} = (2, 4; 2, 4; \dots),$$

$$x'_0 = -1 - \sqrt{2}, \lambda(x'_0)_{\lambda'} = (-3; -2, -4; -2, -4; \dots).$$

$$\lambda' = \frac{1}{4}.$$

$$x_0 = \frac{5 + \sqrt{13}}{6}, \lambda(x_0)_{\lambda'} = (2, 2, 5; 2, 2, 5; \dots),$$

$$x'_0 = -\frac{3 + \sqrt{13}}{2}, \lambda(x'_0)_{\lambda'} = (-4; -2, -2, -5; -2, -2, -5; \dots).$$

$$\lambda' = \frac{1}{5}.$$

$$x_0 = \frac{3 + \sqrt{5}}{4}, \lambda(x_0)_{\lambda'} = (2, 2, 2, 6; 2, 2, 2, 6; \dots),$$

$$x'_0 = -2 - \sqrt{5}, \lambda(x'_0)_{\lambda'} = (-5; -2, -2, -2, -6; -2, -2, -2, -6; \dots).$$

$$\lambda' = \frac{1}{6}.$$

$$x_0 = \frac{7 + \sqrt{29}}{10}, \lambda(x_0)_{\lambda'} = (2, 2, 2, 2, 7; 2, 2, 2, 2, 7; \dots),$$

$$x'_0 = -\frac{5 + \sqrt{29}}{7}, \lambda(x'_0)_{\lambda'} = (-6; -2, -2, -2, -2, -7; -2, -2, -2, -2, -7; \dots).$$



$$\lambda' = \frac{1}{7}.$$

$$x_0 = \frac{4 + \sqrt{10}}{6}, \lambda(x_0)_{\lambda'} = (2, 2, 2, 2, 2, 8; 2, 2, 2, 2, 2, 8; \dots),$$

$$x'_0 = -3 - \sqrt{10}, \lambda(x'_0)_{\lambda'} = (-7; -2, -2, -2, -2, -2, -8; \\ -2, -2, -2, -2, -2, -8; \dots).$$

$$\lambda' = \frac{1}{8}.$$

$$x_0 = \frac{9 + \sqrt{53}}{14}, \lambda(x_0)_{\lambda'} = (2, 2, 2, 2, 2, 2, 9; 2, 2, 2, 2, 2, 2, 9; \dots),$$

$$x'_0 = -\frac{7 + \sqrt{53}}{14}, \lambda(x'_0)_{\lambda'} = (-8; -2, -2, -2, -2, -2, -2, -9; \dots).$$

$$\lambda' = \frac{1}{9}.$$

$$x_0 = \frac{5 + \sqrt{17}}{8}, \lambda(x_0)_{\lambda'} = (2, 2, 2, 2, 2, 2, 2, 10; 2, 2, 2, 2, 2, 2, 2, 10; \dots),$$

$$x'_0 = -4 - \sqrt{17}, \lambda(x'_0)_{\lambda'} = (-9; -2, -2, -2, -2, -2, -2, -2, -10; \dots).$$

$$\lambda' = \frac{1}{d}. \quad d \text{ a positive integer greater than } 1.$$

$$x_0 = \frac{d + 1 + \sqrt{d^2 - 2d + 5}}{2(d - 1)}, \lambda(x_0)_{\lambda'} = (2_1, 2_2, \dots, 2_{d-2}, d + 1; \\ 2_1, 2_2, \dots, 2_{d-2}, d + 1; \dots),$$

$$x'_0 = -\frac{d - 1 + \sqrt{d^2 - 2d + 5}}{2}, \lambda(x'_0)_{\lambda'} = (-d; -2_1, -2_2, \dots, -2_{d-2}, -d - 1; \dots).$$

In the examples just given

$$x'_0 = -\frac{x_0}{x_0 - 1}$$

and it may be observed that the period in  $\lambda(x_0)_{\lambda'}$  is the only one which appears in  $A\lambda(x_0)_{\lambda'}$  with no other change than that of the signs of the elements. When  $d$  is even, the denominator in the expression for  $x'_0$  in terms of  $d$  is 2, but when  $d$  is odd it reduces to 1.

We add some examples where the periods of  $\lambda(x'_0)_{\lambda'}$  and of  $\lambda(x_0)_{\lambda'}$  differ in other respects than merely a change of the signs of the elements.

$$\lambda' = \frac{1}{3}.$$

$$x_0 = \frac{2 + \sqrt{3}}{2}, \lambda(x_0)_{\lambda'} = (4; 4; 4; \dots),$$

$$x'_0 = -\frac{7 + \sqrt{3}}{2}, \lambda(x'_0)_{\lambda'} = (-5; -2, -3; -2, -3; \dots).$$

$$\lambda' = \frac{1}{4}.$$

$$x_0 = \frac{5 + \sqrt{21}}{2}, \lambda(x_0)_{\lambda'} = (5; 5; 5; \dots),$$

$$x'_0 = -\frac{27 + \sqrt{21}}{6}, \lambda(x'_0)_{\lambda'} = (-6; -2, -2, -3; -2, -2, -3; \dots).$$

$$\lambda' = \frac{1}{5}.$$

$$x_0 = \frac{3 + \sqrt{8}}{2}, \lambda(x_0)_{\lambda'} = (6; 6; 6; \dots),$$

$$x'_0 = -\frac{11 + \sqrt{2}}{2}, \lambda(x'_0)_{\lambda'} = (-7; -2, -2, -2, -3; \dots).$$

$$\lambda' = \frac{1}{6}.$$

$$x_0 = \frac{7 + \sqrt{45}}{2}, \lambda(x_0)_{\lambda'} = (7; 7; 7; \dots),$$

$$x'_0 = -\frac{65 + \sqrt{45}}{10}, \lambda(x'_0)_{\lambda'} = (-8; -2, -2, -2, -2, -3; \dots).$$

$$\lambda' = \frac{1}{d}, d \text{ a positive integer greater than one.}$$

$$x_0 = \frac{d+1 + \sqrt{d^2 + 2d - 3}}{2}, \lambda(x_0)_{\lambda'} = (d+1; d+1; d+1; \dots),$$

$$x'_0 = -\frac{(d+1)(2d+1) + \sqrt{d^2 + 2d - 3}}{2(d-1)}, \lambda(x'_0)_{\lambda'} = (-d-2; -2_1, \\ -2_2, \dots, d_{d-2}, -3; \dots).$$

In these examples

$$x'_0 = \frac{dx_0 - d^2 + 1}{-x_0 + d}.$$

Again consider these examples:

$$\lambda' = \frac{1}{3}.$$

$$x_0 = \frac{6 + \sqrt{24}}{3}, \lambda(x_0)_{\lambda'} = (4, 3; 4, 3; 4, 3; \dots),$$

$$x'_0 = -\frac{18 + \sqrt{24}}{5}, \lambda(x'_0)_{\lambda'} = (-5; -3, -2, -3; -3, -2, -3; \dots).$$

$$\lambda' = \frac{1}{4}.$$

$$x_0 = \frac{5 + \sqrt{20}}{2}, \lambda(x_0)_{\lambda'} = (5, 4; 5, 4; 5, 4; \dots),$$

$$x'_0 = -\frac{50 + \sqrt{80}}{11}, \lambda(x'_0)_{\lambda'} = (-6; -2, -3, -2, -2, -3; \dots).$$

$$\lambda = \frac{1}{5}.$$

$$x_0 = \frac{15 + \sqrt{195}}{5}, \quad \lambda(x_0)_{\lambda'} = (6, 5; 6, 5; 6, 5; \dots),$$

$$x'_0 = -\frac{105 + \sqrt{195}}{19}, \quad \lambda(x'_0)_{\lambda'} = (-7, -2, -2, -3, -2, -2, -2, -3;$$

$$\lambda = \frac{1}{d} \text{ } d \text{ a positive integer greater than one.}$$

$$x_0 = \frac{d(d+1) + \sqrt{(d^2+d)(d^2+d-4)}}{2d}, \quad \lambda(x_0)_{\lambda'} = (d+1, d; d+1, d;$$

$$d+1, d; \dots),$$

$$x'_0 = \frac{d(2d^2-d-3) + \sqrt{(d^2+d)(d^2+d-4)}}{2(d^2-d+1)}, \quad \lambda(x'_0)_{\lambda'} = (-d-2; -2_1, -2_2,$$

$$\dots, -2_{d-3}, -3, -2_1, \dots, -2_{d-2}, -3; \dots).$$

In this list as in the preceding

$$x'_0 = \frac{dx_0 - d^2 + 1}{-x_0 + d}.$$

In  $H_I$  and  $H_{II}$  and in the preceding lists the  $\lambda$ -developments ( $\lambda = \lambda'$ ) considered are all periodic.

Let us now consider an example of a non-periodic  $\lambda$ -development ( $\lambda = \lambda'$ ) having an associated  $\lambda$ -development ( $\lambda = \lambda'$ ).

Denote by  $x_0$  the infinite  $\lambda$ -sequence derived from the continued fraction

$$x_{n(2n+1)} = (3_1, 4_1, 4_2, 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4, \dots, 3_1, 3_2, \dots, 3_{2n-1}, 4_1, 4_2, \dots, 4_{2n}),$$

which is a  $\lambda$ -development for  $\lambda' = \frac{1}{4}$ , by taking the limits of both members of this equation as  $n = \infty$ . Then

$$\lambda(x_0)_{\lambda'=\frac{1}{4}} = (3, 4_1, 4_2, 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4, 3_1, 3_2, 3_3, 3_4, 3_5,$$

$$4_1, 4_2, 4_3, 4_4, 4_5, 4_6, \dots).$$

Now let

$$x_0 = -\frac{1}{x_0},$$

then

$$\lambda(x'_0)_{\lambda'=\frac{1}{4}} = (-1, -2, -3, -2, -3, -2; -3, -3, -3; -3, -2,$$

$$-3, -2, -3, -2, -3, -2; -3, -3, -3, -3, -3;$$

$$-3, -2, -3, -2, -3, -2, -3, -2, -3, -2, -3, -2, \dots).$$

It may be remarked that a set of 3's in  $\lambda(x_0)_{\lambda'}$  goes over into a set of -3's in  $A\lambda(x_0)_{\lambda'}$  i. e. in  $\lambda(x'_0)_{\lambda'}$ ; that every 4 in  $\lambda(x'_0)_{\lambda'}$  gives rise to a sequence -3, -2 in  $A\lambda(x_0)_{\lambda'}$ . The semicolons, departing from customary usage, serve to separate



in the associate the sequences arising from sets of 3's in  $\lambda(x_0)_{\lambda'}$  from the sequences arising from sets of 4's.

In the manner indicated any number of non-periodic  $\lambda$ -developments ( $\lambda = \lambda'$ ) having associates may be set up. The property of having associates is not, therefore, characteristic of quadratic irrationalities.

Let it be observed that this process of associated  $\lambda$ -developments ( $\lambda = \lambda'$ ) applies as well to terminating as to infinite  $\lambda$ -developments ( $\lambda = \lambda'$ ).

For example let

$$\lambda(x_0)_{\lambda'=\frac{1}{2}} = (3, 4_1, 4_2, 3_1, 3_2, 3_3, 4_1, 4_2, 4_3, 4_4, \dots, 3_1, 3_2, \dots, \\ 3_{2n-1}, 4_1, 4_2, \dots, 4_{2n}),$$

and let

$$x'_0 = -\frac{1}{x_0}.$$

Then

$$\lambda(x'_0)_{\lambda'=\frac{1}{2}} = (-1, -2; -3_1, -3_2, -3_3, -3, -2, -3, -2, -3, -2, -3, -2, \\ \dots, -3_1, -3_2, \dots, -3_{2n-1}, -3_1, -2_1, -3_2, -2_2, \dots, -3_{2n}, -2_{2n}, -2_{2n}),$$

where the subscript to an element in  $\lambda(x'_0)_{\lambda'}$  indicates that the element is derived from the element with the same subscript in  $\lambda(x_0)_{\lambda'}$ .







